A New Class of Tests of Contagion: Application to Asian Real Estate and Equity Markets

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September 2006

Abstract

A new class of tests of financial contagion which identify transmission channels through changes in higher order moments of returns during financial crises, is proposed. The tests extend the recent class of contagion tests based on changes in correlations, existing tests of coskewness, as well as multivariate normality tests. Applying the framework to test for contagion in real estate and equity markets following the Hong Kong speculative attack in 1997-1998, shows that the contagion coskewness tests identify additional channels that are not identified by the correlation based tests.

Keywords: Contagion testing, coskewness, Lagrange multiplier tests, Asian crisis.

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*We would like to thank Ross Maller and other participants at the FIRN doctoral conference, Sydney, December 2005 for helpful comments. Fry acknowledges funding from ARC grant DP0556371. Corresponding author is Vance Martin, email: vance@unimelb.edu.au.
1 Introduction

An important class of tests of contagion proposed by Forbes and Rigobon (2002), tests for changes in the correlation structure of asset returns between crisis and noncrisis periods adjusted for changes in the volatility of the source country. A significant increase in correlation provides evidence of contagion as it represents additional comovements in asset returns during the crisis period that were not present in the noncrisis period.¹

As correlations are important components of portfolio management, changes in correlations are consistent with contagion arising from the portfolio rebalancing channel emphasised by Flemming, Kirby and Ostdiek (1998) and Kodres and Pritsker (2002). While focusing on correlations is a natural starting point in testing for contagion, it does not however capture fully the risk-return tradeoffs in pricing risk that arise in standard mean-variance models, and definitely does not capture the extensions to higher order moments discussed by Harvey and Siddique (2000).² An important outcome of this class of models is the interaction between the first and the second moments of the joint distribution of returns is measured by coskewness. This suggests that by testing for changes in coskewness during financial crises, additional contagious channels can be detected where investors reevaluate their portfolios through changes in the tradeoff between risk and return.

The aim of this paper is to develop a new class of tests of contagion based on changes in coskewness during a crisis period compared to a noncrisis period. The approach consists of extending existing distributions within the generalised exponential class discussed by Cobb, Koppstein and Chen (1983) and Lye and Martin (1993), that explicitly capture higher order moment interactions. An important property of this class of distributions is that it nests the multivariate normal distribution as a special

¹For additional definitions of contagion, see Dornbusch, Park and Claessens (2000) and Pericoli and Sbracia (2003). For a recent review of alternative tests of contagion, see Dungey, Fry, González-Hermosillo and Martin (2005a, 2005c).

²More recently, Yuan (2005) emphasises the importance of skewness in models of contagion which can arise from information asymmetries and borrowing constraints.
case, thereby providing a framework to construct Lagrange multiplier tests of contagion with the multivariate normal distribution defined under the null. The generality of this framework allows for other tests of contagion to be entertained based on different combinations of higher ordered moments, for example contagion tests of cokurtosis, as well as extending the existing tests of multivariate normality by Mardia (1970), Bera and John (1983) and Richardson and Smith (1993).

The rest of the paper proceeds as follows. Section 2 provides some motivating examples of the importance of coskewness in modelling risk. A key result of these examples is that whilst shocks are assumed to be normal, nonlinearities arising either from risk-return tradeoffs or hedged portfolios containing options, can result in non-normal distributions exhibiting coskewness. The exponential family of distributions is discussed in Section 3, which provides the framework for developing tests of coskewness in Section 4, and tests of contagion in Section 5. These tests are applied in Section 6 to examine channels of contagion in the Asian real estate and equity markets during the Hong Kong crisis in October of 1997. Concluding comments are given in Section 7.

2 Implications of Coskewness in Modelling Risk

Implications of higher order moments for models of risk were initially investigated by Kraus and Litzenberger (1976) and Lee, Moy and Lee (1996), who looked at the effects on beta risk, and more recently by Harvey and Siddique (2000) who extended the mean-variance framework to allow for skewness and by Ang and Chen (2002) who tested for asymmetric correlations. Associated with these models are the recent theoretical contagion models of Kodres and Pritsker (2002) who show the relationship between contagion and risk in a model with information asymmetries, and Yuan (2005) who shows how skewness arises in a model of contagion containing information asymmetries as well as borrowing constraints.
To highlight the relationship between pricing risk and coskewness, two examples are discussed. In the first example, a multivariate GARCH model with in-mean spillover effects arising from risk-return tradeoffs is shown to generate coskewness when shocks are multivariate normal. This model is a multivariate generalisation of the ARCH-M class of models of Engle, Lilien and Robins (1987). The second example looks at the beta risk of a portfolio which includes options written on assets in the market portfolio. The inclusion of options in the portfolio introduces nonlinearities whereby the beta risk is a function of coskewness even though returns are normally distributed.

2.1 MGARCH Models of Risk Spillovers

Consider the following multivariate GARCH model of asset returns \( (r_{1,t}, r_{2,t}) \), where the variance of the first asset enters the mean of the second asset.\(^3\) In this model the risk of the second asset is priced according to the “spillover risk” from the first asset. The model is specified as

\[
\begin{align*}
    r_{1,t} &= \sqrt{h_{1,t}} u_{1,t} \\
    r_{2,t} &= \mu h_{1,t} + \sqrt{h_{2,t}} u_{2,t} \\
    h_{1,t} &= 0.5 + 0.2 h_{1,t-1} u_{1,t-1}^2 + 0.6 h_{1,t-1} \\
    h_{2,t} &= 0.5 + 0.2 h_{2,t-1} u_{2,t-1}^2 + 0.6 h_{2,t-1} \\
    u_{i,t} &\sim N(0, \sigma_i^2), \quad i = 1, 2 \\
    E[u_{1,t} u_{2,t}] &= 0,
\end{align*}
\]

where \( r_{i,t} \) is the return on asset \( i \), with GARCH(1,1) conditional volatility \( h_{i,t} \). The parameter \( \kappa \) controls the size of the spillover risk from asset 1 to asset 2, and hence controls the risk-return tradeoff of asset 2. Setting \( \kappa = 0.5 \), yields a linear relationship between the expected return \( (\mu h_{1,t}^\kappa) \) and the risk \( (\sqrt{h_{1,t}}) \). For values of \( \kappa > 0.5 \), increases in risk result in the expected return increasing at an increasing rate. For

\(^3\)For simplicity, the model assumes that the conditional variances are independent. More general MGARCH specifications can be considered, including the dynamic conditional correlation (DCC) model of Engle (2002).
values of $0 < \kappa < 0.5$, the expected return increases at a decreasing rate for increases in risk. These three cases are referred to as risk aversion ($\kappa > 0.5$), risk neutrality ($\kappa = 0.5$) and risk seeking ($0 < \kappa < 0.5$).

To highlight the properties of the model, (1) is simulated for $T = 100,000$ observations with $\mu = 2$, and for the following values of $\kappa$

$$\kappa = \{1.0, 0.5, 0.2, 0.0\}.$$  \hspace{1cm} (2)

For each simulation run the following statistic is computed

$$\hat{\psi}(r_{1,t-k}^m, r_{2,t-l}^n) = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{r_{1,t-k} - \hat{\mu}_1}{\hat{\sigma}_1} \right)^m \left( \frac{r_{2,t-l} - \hat{\mu}_2}{\hat{\sigma}_2} \right)^n,$$  \hspace{1cm} (3)

for various values of the power indices $m$ and $n$, and the lag indices $k$ and $l$, where $\hat{\mu}_i$ and $\hat{\sigma}_i$ are respectively the sample mean and standard deviation of the simulated returns of asset $i$. The indices $(m, n)$ control the type of statistic being computed where $m = n$ yields the correlation between the two returns, while setting $m = 2, n = 1$, or $n = 1, m = 2$, yields the coskewness between the two returns. The indices $(k, l)$ allow for dynamic relationships with $k = l = 0$ representing contemporaneous relations, and with $k > 0$ or $l > 0$ representing dynamic relations.

The sampling distributions of (3) from simulating (1) 10,000 times for each value of $\kappa$ in (2), are given in Figure 1.\textsuperscript{5} This figure contains nine graphs which differ according to the power indices $(m, n)$ and the dynamic indices $(k, l)$. Within each sub-figure there are four sampling distributions corresponding to the four values of $\kappa$ in (2). A value of $\kappa = 0$, is interpreted as the noncrisis period, which corresponds to returns in (1) being independent and thus represents the benchmark case in the simulation experiment. The crisis period is identified by spillover risk with the risk-return tradeoff determined by $\kappa > 0$.\textsuperscript{6}

\textsuperscript{4}The model is simulated with an additional 100 observations at the start of the sample which are then excluded to circumvent startup problems with the imposition of initial conditions.

\textsuperscript{5}The sampling distributions are computed using a nonparametric normal kernel with a rule of thumb bandwidth.

\textsuperscript{6}Extensions of the MGARCH model which take into account spillovers in the conditional vari-
The most striking result is in Figure 1(f) which shows that the crisis period is characterised by an increase in coskewness between \( r_{2,t} \) and \( r_{1,t-1}^2 \), which progressively increases in significance for higher values of \( \kappa \). The fact that the direction of coskewness is determined by the effects of volatility in asset 1 on the expected return on asset 2, is supported by Figure 1(3) which shows that by reversing the order of coskewness the statistic becomes independent of \( \kappa \). Figure 1(c) also shows some evidence of an increase in correlation during the crisis period, albeit weaker evidence than that obtained from the coskewness statistics. All other sampling distributions in the crisis period show little difference from the respective noncrisis sampling distributions. In particular, there is no change in the sampling distributions in the crisis and noncrisis periods when the distributions are based on the lag of \( r_{2,t} \).

The results of the sampling experiment in Figure 1 highlight the potential importance of measuring coskewness amongst asset returns as a way to identify changes in risk preferences during financial crises. These results also emphasise the importance of modelling dynamics to identify the source of shocks and in doing so, provide a way of identifying causal linkages amongst higher order moments.

### 2.2 Beta Risk

The beta of a portfolio is defined as

\[
\beta_{\text{eta}} = \frac{\text{Cov}(r_{p,t}, r_{m,t})}{\text{Var}(r_{m,t})},
\]

where \( r_{p,t} \) is the return on the portfolio and \( r_{m,t} \) is the return on the market. For simplicity, let the market portfolio consist of two assets with returns \( r_{1,t} \) and \( r_{2,t} \), which are normally distributed with zero means, variances \( \sigma_1^2 \) and covariance \( \sigma_{1,2} \). The return on the market is

\[
r_{m,t} = \phi_1 r_{1,t} + \phi_2 r_{2,t},
\]

iances across the two asset markets could also generate coskewness following the modelling strategy of Glosten, Jagannathan and Runkle (1993).
Figure 1: Sampling distributions of (3) from the bivariate MGARCH model for alternative values of the risk aversion parameter $\kappa$. Panel (a) is the sampling distribution of the correlation coefficient, (b) and (c) are the sampling distributions of the cross autocorrelation statistics, while the remaining graphs give the sampling distributions of contemporaneous and dynamic coskewness statistics.
where $\phi_i$ is the weight on asset $i$. By assumption, $E[r_{m,t}] = 0$ and $E[r_{m,t}^2] = \phi_1^2 \sigma_1^2 + \phi_2^2 \sigma_2^2 + 2\phi_1 \phi_2 \sigma_{1,2}.

Now suppose that the portfolio just includes options written on both assets in the market portfolio. The return on the portfolio is (Hull, 2000)

$$r_{p,t} = \alpha_1 r_{1,t} + \alpha_2 r_{2,t} + \beta_{1,1} r_{1,t}^2 + \beta_{2,2} r_{2,t}^2 + \beta_{1,2} r_{1,t} r_{2,t}. \quad (6)$$

The weights in the portfolio $(\alpha_i, \beta_{i,j})$ can be chosen to make the portfolio delta neutral $\alpha_i = \delta_i$, where $\delta_i$ represents the delta, and gamma neutral $\beta_{i,j} = \gamma_{i,j}$ where $\gamma_{i,i}$ represents the gamma and $\gamma_{i,j}$ the “cross gamma”.

The inclusion of options in the portfolio causes odd-order higher moments between the returns on the portfolio and the market to be nonzero even though the underlying returns are normal. For example, consider the coskewness of the two returns $E[r_{p,t}^2] = 0$. Focussing on the first term in the numerator

$$\psi \left( r_{p,t}^1, r_{m,t}^2 \right) = \frac{E \left[ (r_{p,t} - E[r_{p,t}]) (r_{m,t} - E[r_{m,t}])^2 \right]}{\sqrt{E \left[ (r_{p,t} - E[r_{p,t}])^2 \right] E \left[ (r_{m,t} - E[r_{m,t}])^2 \right]}}, \quad (7)$$

as $E[r_{m,t}] = 0$. Focussing on the first term in the numerator

$$E[r_{p,t} r_{m,t}^2] = E \left[ (\alpha_1 r_{1,t} + \alpha_2 r_{2,t} + \beta_{1,1} r_{1,t}^2 + \beta_{2,2} r_{2,t}^2 + \beta_{1,2} r_{1,t} r_{2,t}) \left( \phi_1 r_{1,t} + \phi_2 r_{2,t} \right)^2 \right]$$

$$= \beta_{1,1} \left[ 3\phi_1^2 \sigma_1^2 + \phi_2^2 \sigma_1^2 \sigma_2^2 (1 + 2\rho^2) + 6\phi_1 \phi_2 \rho \sigma_1^2 \sigma_2 \right]$$

$$+ \beta_{2,2} \left[ \phi_1^2 \sigma_1^2 \sigma_2^2 (1 + 2\rho^2) + 3\phi_2^2 \sigma_2^4 + 6\phi_1 \phi_2 \rho \sigma_1^2 \sigma_2^3 \right]$$

$$+ \beta_{1,2} \left[ 3\phi_1^2 \rho \sigma_1^2 \sigma_2 + 3\phi_2^2 \rho \sigma_1^2 \sigma_2^3 + 2\phi_1 \phi_2 \sigma_1^2 \sigma_2^2 (1 + 2\rho^2) \right], \quad (8)$$

where the moment properties of the bivariate normal distribution in Appendix A are used. The second term in the numerator is

$$E[r_{p,t}] E[r_{m,t}^2] = (\beta_{1,1} \sigma_1^2 + \beta_{2,2} \sigma_2^2 + \beta_{1,2} \sigma_{1,2}) \left( \phi_1^2 \sigma_1^2 + \phi_2^2 \sigma_2^2 + 2\phi_1 \phi_2 \sigma_{1,2} \right). \quad (9)$$

Upon substituting the expression in (8) and (9) for the first and second terms into the numerator in (7) shows that coskewness is

$$\psi \left( r_{p,t}^1, r_{m,t}^2 \right) \neq 0,$$
which is true even though the underlying distributions of the returns are normal. This result suggests that the \( \beta \) as defined in (4), does not correctly price all of the risks when a portfolio contains options as it ignores the higher order moments effects.

3 The Exponential Family of Distributions

The previous section highlights the potential importance of testing for coskewness to detect contagion through portfolio rebalancing and modelling risk in general. In this section a framework is proposed for developing tests of coskewness which is based upon extending the existing set of subordinate distributions within the generalised exponential class.

Define \( r = \{r_1, r_2, \ldots, r_K\} \) as a \( K \) dimensional vector of random variables. The multivariate generalised exponential family of distributions is given by

\[
f(r) = \exp \left( \sum_{i=1}^{M} \theta_i g_i(r) - \eta \right),
\]

where \( \theta = \{\theta_1, \theta_2, \ldots, \theta_M\} \) is a \( M \) dimensional vector of parameters, \( g_i(r) \) is an arbitrary function of the \( K \) dimensional random variable \( r \), and \( \eta \) is a normalising constant defined as

\[
\eta = \ln \int \cdots \int \exp \left( \sum_{i=1}^{M} \theta_i g_i(r) \right) dr_1 dr_2 \cdots dr_K,
\]

which ensures that \( f(r) \) is a well defined probability distribution

\[
\int \cdots \int \exp \left( \sum_{i=1}^{M} \theta_i g_i(r) - \eta \right) dr_1 dr_2 \cdots dr_K = 1.
\]

Typical choices for \( g_i(r) \) in (10) are polynomials and cross-products in the elements of \( r \). For example, setting \( g_i(r) = r_i^2 \) yields

\[
f(r) = \exp \left( \sum_{i=1}^{M} \theta_i r_i^2 - \eta \right),
\]

which is the \( M \) dimensional independent normal distribution. Setting \( M = 3 \) and choosing \( g_1(r) = r_1^2, g_2(r) = r_2^2 \), and \( g_3(r) = r_1 r_2 \), yields the bivariate normal distribu-
for the case of zero means. For this formulation of the generalised distribution, the parameters $\theta_1$ and $\theta_2$ control the respective variances, whilst $\theta_3$ controls the degree of dependence between $r_1$ and $r_2$. In the usual parameterisation of the bivariate normal distribution the degree of dependence is controlled by the correlation parameter $\rho$, which is clearly related to $\theta_3$.

The examples in (12) and (13) suggest that generalisations of the multivariate normal class of distributions are obtained by defining the $g_i (r)$ functions in terms of powers greater than two. For example, the univariate ($K = 1$) generalised normal distribution discussed by Cobb, Koppstein and Chen (1983) and Lye and Martin (1993) is given by

$$ f (r) = \exp \left( \theta_1 r_1^2 + \theta_2 r_2^2 + \theta_3 r_1 r_2 - \eta \right), \quad (14) $$

which is (10) with $M = 4$ and $g_i (r) = r_i^i$. These extensions of this class of bivariate normal distributions would appear to be natural choices in the area of finance as they provide explicit expressions for higher order moments, including skewness and kurtosis.\(^7\)

The univariate generalised normal distribution in (14) suggests that a natural way of extending this model to a bivariate distribution would be to include various higher order cross-moment effects including coskewness and cokurtosis terms. For example, consider the following bivariate ($K = 2$) generalised normal distribution with $M = 6$ in (10)

$$ f (r) = \exp \left( \theta_1 r_1^2 + \theta_2 r_2^2 + \theta_3 r_1 r_2 + \theta_4 r_1^3 r_2 + \theta_5 r_2^3 r_1 + \theta_6 r_1^2 r_2^2 - \eta \right). \quad (15) $$

The terms $r_1 r_2^2$ and $r_1^2 r_2$ represent two measures of coskewness between $r_1$ and $r_2$ which are controlled by the parameters $\theta_4$ and $\theta_5$ respectively, while the term $r_1^2 r_2^2$ is suggested by the focus on generalising the multivariate normal distribution, multivariate generalisations of the Student $t$ class of distributions investigated by Lye and Martin (1993), can also be entertained.

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\(^7\)Whilst the focus is on generalising the multivariate normal distribution, multivariate generalisations of the Student $t$ class of distributions investigated by Lye and Martin (1993), can also be entertained.
represents cokurtosis between $r_1$ and $r_2$, which is controlled by the parameter $\theta_6$.\(^8\) This distribution allows for three levels of dependence. The first is the usual channel which is controlled by the parameter $\theta_3$, effectively the correlation parameter. The second channel is through the parameters $\theta_4$ and $\theta_5$ where dependence is captured by the interaction between the first moment of asset 1 ($r_1$) and the second moment of asset 2 ($r_2^2$), namely coskewness. The third channel is controlled by the cokurtosis parameter $\theta_6$, which allows for the interaction of the second moments of the two random variables.

Further extensions of (15) can be contemplated which allow for higher order moment interaction terms. For the rest of this paper, the focus is on distributions of the form of (15) with special emphasis given to modelling and testing coskewness.

\section{Testing Moments}

The general exponential class of distributions discussed above suggests a framework for developing higher order moment tests including coskewness, where the model under the null hypothesis is multivariate normality. As it is easier to estimate a normal distribution (the constrained model) than the generalised normal distribution (the unconstrained model), this suggests that constructing Lagrange multiplier tests is a convenient framework to develop tests of contagion. Furthermore, as the multivariate normal distribution is a special case, this suggests that the multivariate tests of normality proposed by Mardia (1970), Bera and John (1983) and Richardson and Smith (1993), can be generalised.

\textbf{Theorem 1} Let $r$ be an iid random variable of dimension $K$ with the generalised exponential distribution
\[
    f(r) = \exp(h - \eta),
\]
where $h = \sum_{i=1}^{M} \theta_i g_i(r)$, $\theta$ is a $M$ vector of parameters summarising the moments of the distribution and $\eta$ is the normalising constant defined in (11). The information

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\(^8\)For this distribution to have finite moments of all orders, it is required that $\theta_6 < 0$. 

11
The information matrix is given by

$$I = T \left( E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial h}{\partial \theta'} \right] \right),$$

(17)

where $\ln L_t = \ln f (r_t)$ represents the log of the likelihood at the $t^{th}$ observation and $T$ is the sample size.

**Proof.** The log of the likelihood at observation $t$ is

$$\ln L_t = h_t - \eta,$$

(18)

where $h_t = \sum_{i=1}^M \theta_i g_i (r_t)$, while the first and second derivatives are respectively

$$\frac{\partial \ln L_t}{\partial \theta} = \frac{\partial h_t}{\partial \theta} - \frac{\partial \eta}{\partial \theta},$$

(19)

$$\frac{\partial^2 \ln L_t}{\partial \theta \partial \theta'} = \frac{\partial^2 h_t}{\partial \theta \partial \theta'} - \frac{\partial^2 \eta}{\partial \theta \partial \theta'}.$$

(20)

Taking expectations of the second derivative and changing the sign yields the information matrix at $t$

$$I_t = -E \left[ \frac{\partial^2 \ln L_t}{\partial \theta \partial \theta'} \right] = \frac{\partial^2 \eta}{\partial \theta \partial \theta'} - E \left[ \frac{\partial^2 h_t}{\partial \theta \partial \theta'} \right].$$

(21)

This expression is simplified by differentiating

$$\eta = \ln \int \exp (h) \, dr,$$

which gives

$$\frac{\partial \eta}{\partial \theta} = \frac{\int \frac{\partial h}{\partial \theta} \exp [h] \, dr}{\int \exp [h] \, dr} = E \left[ \frac{\partial h}{\partial \theta} \right].$$

(22)
Differentiating a second time gives

$$\frac{\partial^2 \eta}{\partial \theta \partial \theta'} = \left( \int \frac{\partial^2 h}{\partial \theta \partial \theta'} \exp[h] \, dr + \int \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \exp[h] \, dr \right) \left( \int \exp[h] \, dr \right)^{-2} - \left( \int \frac{\partial h}{\partial \theta} \exp[h] \, dr \right) \left( \int \frac{\partial h}{\partial \theta'} \exp[h] \, dr \right) \left( \int \exp[h] \, dr \right)^{-2}$$

$$= E \left[ \frac{\partial^2 h}{\partial \theta \partial \theta'} \right] + E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial^2 h}{\partial \theta \partial \theta'} \right] \quad (23)$$

Substituting this expression into the information matrix in (21) yields

$$I_t = E \left[ \frac{\partial^2 h}{\partial \theta \partial \theta'} \right] + E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial h}{\partial \theta'} \right]$$

$$= E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial h}{\partial \theta'} \right], \quad (24)$$

where the last step uses the property that $r$ is iid, so $E \left[ \frac{\partial^2 h}{\partial \theta \partial \theta'} \right] = E \left[ \frac{\partial^2 h}{\partial \theta' \partial \theta} \right]$. Finally,

$$I = \sum_t I_t$$

$$= T \left( E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial h}{\partial \theta'} \right] \right),$$

from the iid assumption. ■

The advantage of (24) is that it is not necessary to use the normalising constant $\eta$, to derive the information matrix. Instead, the information matrix can be conveniently derived simply by taking expectations of the functions of the first derivatives of $h$. Having derived the information matrix, the Lagrange multiplier statistic is given by

$$LM = G' I^{-1} G, \quad (25)$$

where

$$G = \frac{\partial \ln L_t}{\partial \theta'} \bigg|_{\theta = \theta_0}, \quad (26)$$
is the gradient vector of the log of the likelihood and

\[ I = \left. \frac{\partial \ln L_t}{\partial \theta \partial \theta'} \right|_{\theta = \theta_0}, \tag{27} \]

is the information matrix, which are both evaluated under the null \( \theta = \theta_0 \).

**Example 1: Coskewness Test**

Consider the following generalisation of the bivariate normal distribution given in (13)

\[
f(r_{1,t}, r_{2,t}) = \exp \left[ -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 
- 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) 
+ \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] dr_1 dr_2, \tag{28} \]

where

\[
\eta = \ln \int \int \exp \left[ -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 
- 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) 
+ \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] dr_1 dr_2
= \ln \int \int \exp [h] dr_1 dr_2, \tag{29} \]

and

\[
h = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 
- 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) 
+ \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2. \tag{30} \]

A test of the restriction

\[ H_0 : \phi = 0, \tag{31} \]
constitutes a test of coskewness. Under the null hypothesis, the distribution is bivariate normal where the maximum likelihood estimators of the unknown parameters are

\[
\tilde{\mu}_i = \frac{1}{T} \sum_t r_{i,t}; \tilde{\sigma}_i^2 = \frac{1}{T} \sum_t (r_{i,t} - \tilde{\mu}_i)^2; \tilde{\rho} = \frac{1}{T} \sum_t \left( \frac{r_{1,t} - \tilde{\mu}_1}{\tilde{\sigma}_1} \right) \left( \frac{r_{2,t} - \tilde{\mu}_2}{\tilde{\sigma}_2} \right).
\]  

(32)

The Lagrange multiplier statistic is (see Appendix C.1 for details)

\[
LM = G' I^{-1} G \\
= \frac{T}{4\rho^2 + 2} \left[ \frac{1}{T} \sum_{t=1}^T \left( \frac{r_{1,t} - \tilde{\mu}_1}{\tilde{\sigma}_1} \right) \left( \frac{r_{2,t} - \tilde{\mu}_2}{\tilde{\sigma}_2} \right) \right]^2.
\]  

(33)

Under the null, \(LM\) is distributed asymptotically as

\[
LM \xrightarrow{d} \chi^2_1.
\]  

(34)

**Example 2: Coskewness and Dependence Joint Test**

From (28), it follows that a joint test of coskewness and dependence is based on the null hypothesis

\[
H_0 : \phi = 0, \rho = 0.
\]  

(35)

Under \(H_0\) the distribution is an independent bivariate normal distribution. The Lagrange multiplier statistic is (see Appendix C.2 for details)

\[
LM = G' I^{-1} G \\
= T \left\{ \frac{1}{T} \sum_{t=1}^T \left( \frac{r_{1,t} - \tilde{\mu}_1}{\tilde{\sigma}_1} \right) \left( \frac{r_{2,t} - \tilde{\mu}_2}{\tilde{\sigma}_2} \right) \right\}^2 + \frac{1}{2} \left[ \frac{1}{T} \sum_{t=1}^T \left( \frac{r_{1,t} - \tilde{\mu}_1}{\tilde{\sigma}_1} \right) \left( \frac{r_{2,t} - \tilde{\mu}_2}{\tilde{\sigma}_2} \right)^2 \right]^2.
\]  

(36)

which is asymptotically distributed as \(\chi^2_2\) under the null hypothesis.

## 5 Contagion Testing

The aim of the correlation based tests of contagion by Forbes and Rigobon (2002) is to identify whether a shock to the returns of one asset market has a different impact on
the level of returns in another market during a financial crisis compared to a noncrisis period. From the discussion of Section 2, a potentially more important transmission mechanism is one that captures the spillover risk across asset markets which changes the risk-return tradeoffs. This suggests that the important higher order moment interactions are between the mean and the variance of the distribution, as measured by coskewness. For this reason the new class of tests of financial market contagion proposed focus on changes in coskewness, although extensions to other higher order moments including cokurtosis, are also briefly mentioned.

Prior to deriving the coskewness version of the contagion test, the change in the correlation test proposed by Forbes and Rigobon (2002), together with some variations of this test suggested by Dungey, Fry, González-Hermosillo and Martin (2005b), are presented initially. In doing so, some further extensions of the correlation based tests of contagion are also proposed which take into account further corrections of the asymptotic distribution of the test statistic. These extensions of the correlation tests also motivate alternative forms of the coskewness based contagion tests.

In deriving the contagion tests the following notation is used. The noncrisis period is denoted as \( x \), the crisis period as \( y \) and the total period as \( z \). The correlation between the two asset returns is denoted as \( \rho_x \) (noncrisis), \( \rho_y \) (crisis) and \( \rho_z \) (total). The sample sizes of the noncrisis, crisis and total periods are respectively \( T_x, T_y \) and \( T_z = T_x + T_y \).

### 5.1 Correlation Change Tests

Consider testing for contagion from a source asset market, market \( i \), to a recipient market, market \( j \). The Forbes and Rigobon (2002) statistic \( (FR_1) \) to test for contagion
from \(i\) to \(j\), is\(^9\)

\[
FR_1 \left( i \rightarrow j \right) = \left( \frac{1}{2} \ln \left( \frac{1 + \hat{\rho}_{y|z_i}}{1 - \hat{\rho}_{y|z_i}} \right) - \frac{1}{2} \ln \left( \frac{1 + \hat{\rho}_z}{1 - \hat{\rho}_z} \right) \right)^2,
\]

(37)

where

\[
\hat{\rho}_{y|z_i} = \frac{\hat{\rho}_y}{\sqrt{1 + \left( \frac{s_{y,i}^2 - s_{z,i}^2}{s_{z,i}^2} \right) (1 - \hat{\rho}_y^2)}},
\]

(38)

represents the heteroskedastic adjusted sample correlation coefficient that takes into account increases in volatility in the source market (market \(i\)) during the crisis period relative to the total period. The remaining quantities are \(\hat{\rho}_z\), the sample correlation coefficient during the total period, and \(s_{y,i}^2\) and \(s_{z,i}^2\), the sample variances of asset returns in country \(i\) during the crisis and total periods respectively. This statistic is simply a test of the difference in correlations having adjusted the crisis correlation for any changes in the volatility of the source market asset returns. Following Forbes and Rigobon, the statistic is transformed using the Fisher transformation (Kendall and Stuart (1969), p.390-391) to improve the finite sample properties of the statistic. Under the null hypothesis of no contagion, Forbes and Rigobon assume that

\[
FR_1 \left( i \rightarrow j \right) \xrightarrow{d} \chi^2_1.
\]

(39)

Large values of the test statistic is evidence of a significant change in the comovements between asset returns during a crisis period.

A potential problem with using (37) to test for contagion is that it is based on the assumption that the correlation coefficients from the crisis and total sample periods are independent, which is clearly not correct given that the periods overlap. Two solutions

\(^9\)The focus here is to identify changes in the distributions of returns across periods. The statistic \(FR_1\) is actually the square of the statistic suggested by Forbes and Rigobon (2002), thereby making this a two-sided test. Using this form of the statistic means that it is not possible to distinguish between contagion and flight to quality, where the former (latter) is represented by an increase (decrease) in correlation. Of course these two hypotheses can be accommodated by not computing the “square” version of the test and performing one-sided tests.
are adopted. The first is to perform the test by comparing the crisis period adjusted correlation with the noncrisis period correlation. The test statistic in this case is

\[ FR_2 (i \rightarrow j) = \frac{(\hat{\nu}_{y|x_i} - \hat{\rho}_x)^2}{Var (\hat{\nu}_{y|x_i} - \hat{\rho}_x)} , \tag{40} \]

where

\[ \hat{\nu}_{y|x_i} = \frac{\hat{\rho}_y}{\sqrt{1 + \left( \frac{s^2_{y,i} - s^2_{x,i}}{s^2_{x,i}} \right) (1 - \hat{\rho}_y^2)}} \tag{41} \]

has the same interpretation as (38) except that now the increase in volatility of the source country is measured with respect to the volatility of country 1 during the noncrisis period, \( s^2_{x,i} \) is the sample variance of asset returns in market 1 during the noncrisis period, and \( \hat{\rho}_x \) is the sample correlation coefficient during the noncrisis period. The pertinent variance is given by (see Appendix C.3 for details)

\[ Var (\hat{\nu}_{y|x_i} - \hat{\rho}_x) = Var (\hat{\nu}_{y|x_i}) + Var (\hat{\rho}_x) - 2Cov (\hat{\nu}_{y|x_i}, \hat{\rho}_x) , \tag{42} \]

where

\[ Var (\hat{\nu}_{y|x_i}) = \frac{1}{2 \left(1 + \delta \right)} \left( \frac{(1 + \delta)^2}{1 + \delta (1 - \rho_y^2)} \right)^3 \left[ \frac{1}{T_y} \left( (2 - \rho_y^2) (1 - \rho_y^2)^2 \right) + \frac{1}{T_x} \left( \rho_y^2 (1 - \rho_y^2)^2 \right) \right] , \]

\[ Var (\hat{\rho}_x) = \frac{1}{T_x} (1 - \rho_x^2)^2 , \tag{43} \]

\[ Cov (\hat{\nu}_{y|x_i}, \hat{\rho}_x) = \frac{1}{2 T_x} \rho_y \rho_x \left( 1 - \rho_y^2 \right) (1 - \rho_x^2) (1 + \delta) \sqrt{1 + \delta (1 - \rho_y^2)} \right] , \]

and

\[ \delta = \left( \frac{\sigma^2_{y,i} - \sigma^2_{x,i}}{\sigma^2_{x,i}} \right) \tag{44} \]

which is the proportionate change in the volatility of returns in the source market, market \( i \). This statistic is computed by replacing the population quantities in (43) and (44) by their respective sample estimates.

An important special case of the variance expression in (42) occurs under the assumption of independence in both periods, \( \rho_x = \rho_y = 0 \), in which case (42) reduces
to

$$\text{Var} \left( \hat{\nu}_{y|x}, \hat{\rho}_z \right) = \frac{1}{(1 + \delta) T_y} + \frac{1}{T_x}. \tag{1 + \delta} T_y + 1 T_x \text{.}$$

A further simplification is achieved under the additional assumption of no increase in the volatility in the source market ($\delta = 0$)

$$\text{Var} \left( \hat{\nu}_{y|x}, - \hat{\rho}_x \right) = \frac{1}{T_y} + \frac{1}{T_x}, \tag{1 + \delta} T_y + 1 T_x,$$

which is just the usual asymptotic formula for the variance of the difference in two (independent) sample correlation coefficients.

A second strategy to correct $FR_1$ is to rederive the asymptotic standard error in (37) that takes into account the sampling dependence across the two periods. From (42)

\begin{align*}
\text{Var} \left( \hat{\nu}_{y|z}, \hat{\rho}_z \right) &= \text{Var} \left( \hat{\nu}_{y|z} \right) + \text{Var} \left( \hat{\rho}_z \right) - 2 \text{Cov} \left( \hat{\nu}_{y|z}, \hat{\rho}_z \right) \\
&= \text{Var} \left( \hat{\nu}_{y|z} \right) + \text{Var} \left( \hat{\rho}_z \right) - 2 \text{Var} \left( \hat{\rho}_z \right) \\
&= \text{Var} \left( \hat{\nu}_{y|z} \right) - \text{Var} \left( \hat{\rho}_z \right) \\
&= \frac{1}{T_y} - \frac{1}{T_z}, \tag{45}
\end{align*}

where the second step follows Dungey, Fry, González-Hermosillo and Martin (2005c), while the last step uses the results of the multivariate normal distribution.\(^{10}\) Using (45) as the variance in (37), the third version of the Forbes and Rigobon statistic is then

\begin{align*}
FR_3 \left( i \to j \right) &= \left( \frac{1}{2} \ln \left( \frac{1 + \hat{\nu}_{y|z}}{1 - \hat{\nu}_{y|z}} \right) - \frac{1}{2} \ln \left( \frac{1 + \hat{\rho}_z}{1 - \hat{\rho}_z} \right) \right)^2 \\
&\quad \sqrt{\frac{1}{T_y - 3} - \frac{1}{T_z - 3}}. \tag{46}
\end{align*}

As the standard error in (37) ignores the negative covariance in (45) the statistic in (37) is biased towards zero, resulting in the size of the test having a smaller Type I

\(^{10}\)For further details, see Dungey, Fry, González-Hermosillo and Martin (2005c). This expression is valid for the case when $\rho = 0$ and should serve as a good approximation to the asymptotic variance for small values of $\rho$. For larger departures of $\rho$ from zero, the expression for the asymptotic variance is more involved.
error compared to $FR_2$ and $FR_3$. Using standard conventional significance levels this means that there is a much smaller probability of finding evidence of contagion when the test of contagion is based on (37); a result which is consistent with the empirical results of Forbes and Rigobon (2002) who do not find evidence of contagion in their empirical applications.

5.2 Coskewness Change Tests

The coskewness test of contagion is based on identifying significant changes in coskewness between the crisis period and a noncrisis period. Analogous to the correlation based tests of contagion, alternative versions of the test are presented depending on the choice of the noncrisis period and whether the assumption of independence of the sample statistics across sample periods is invoked. In addition, within each of these versions of the test, there are two additional variants which depend on whether the source country coincides with the level or the square.

In the first version of the coskewness test for contagion, the comparison is based on the crisis and total sample periods under the (incorrect) assumption that the sample statistics are not correlated. To test for contagion from market $i$ to market $j$, the contagion coskewness statistics are

\[
CS_1 (i \to j; r_i^1, r_j^2) = \left( \frac{\hat{\psi}_y (r_i^1, r_j^2) - \hat{\psi}_z (r_i^1, r_j^2)}{\sqrt{4\hat{\nu}_y|z_i + 2 + 4\hat{\rho}_z^2 + 2T}} \right)^2, \tag{47}
\]

\[
CS_1 (i \to j; r_i^2, r_j^1) = \left( \frac{\hat{\psi}_y (r_i^2, r_j^1) - \hat{\psi}_z (r_i^2, r_j^1)}{\sqrt{4\hat{\nu}_y|z_i + 2 + 4\hat{\rho}_z^2 + 2T}} \right)^2, \tag{48}
\]
where

\[
\hat{\psi}_y (r_i^m, r_j^n) = \frac{1}{T_y} \sum_{t=1}^{T_y} \left( \frac{y_i,t - \hat{\mu}_{y_i}}{\sigma_{y_i}} \right)^m \left( \frac{y_j,t - \hat{\mu}_{y_j}}{\sigma_{y_j}} \right)^n,
\]

\[
\hat{\psi}_x (r_i^m, r_j^n) = \frac{1}{T_x} \sum_{t=1}^{T_x} \left( \frac{z_i,t - \hat{\mu}_{z_i}}{\sigma_{z_i}} \right)^m \left( \frac{z_j,t - \hat{\mu}_{z_j}}{\sigma_{z_j}} \right)^n.
\]

The source country is identified by the term \( \hat{\nu}_{y|x} \), which replaces the pertinent correlation coefficient in (33) by the Forbes-Rigobon adjusted correlation coefficient used above. In (47) the contagion channel tested is from a shock in the level of the returns of the source country on the volatility of returns of the recipient country. In (48) the roles are reversed with the contagion channel being tested corresponding to a shock in the volatility of the returns of the source country on the level of returns of the recipient country. This second form of the test corresponds to the MGARCH model specification in Section 2.

The second version of the contagion coskewness test is based on comparing the coskewness statistics from the crisis and noncrisis sample periods

\[
CS_2 (i \rightarrow j; r_i^1, r_j^2) = \left( \frac{\hat{\psi}_y (r_i^1, r_j^2) - \hat{\psi}_x (r_i^1, r_j^2)}{4\hat{\nu}_{y|x|i} + 2\frac{T_y}{T_y} + 4\hat{\rho}_{y|x}^2 + 2} \right)^2,
\]

\[
CS_2 (i \rightarrow j; r_i^2, r_j^1) = \left( \frac{\hat{\psi}_y (r_i^2, r_j^1) - \hat{\psi}_x (r_i^2, r_j^1)}{4\hat{\nu}_{y|x|i} + 2\frac{T_y}{T_y} + 4\hat{\rho}_{y|x}^2 + 2} \right)^2,
\]

where

\[
\hat{\psi}_y (r_i^m, r_j^n) = \frac{1}{T_y} \sum_{t=1}^{T_y} \left( \frac{y_i,t - \hat{\mu}_{y_i}}{\sigma_{y_i}} \right)^m \left( \frac{y_j,t - \hat{\mu}_{y_j}}{\sigma_{y_j}} \right)^n,
\]

\[
\hat{\psi}_x (r_i^m, r_j^n) = \frac{1}{T_x} \sum_{t=1}^{T_x} \left( \frac{z_i,t - \hat{\mu}_{z_i}}{\sigma_{z_i}} \right)^m \left( \frac{z_j,t - \hat{\mu}_{z_j}}{\sigma_{z_j}} \right)^n.
\]

The third version of the coskewness contagion test is based on comparing the coskewness statistics from the crisis and total sample periods, as for \( FR_3 \), but cor-
recting the asymptotic variance to allow for the dependence in the statistics arising
from overlapping data. Using an analogous expression to (45) for the coskewness coef-
ficients, the statistics are

\[
CS_3 (i \rightarrow j; r^1_i, r^2_j) = \frac{\left( \hat{\psi}_y (r^1_i, r^2_j) - \hat{\psi}_z (r^1_i, r^2_j) \right)^2}{\sqrt{\text{Var} \left( \hat{\psi}_y (r^1_i, r^2_j) - \hat{\psi}_z (r^1_i, r^2_j) \right)}},
\]

(51)

\[
CS_3 (i \rightarrow j; r^2_i, r^1_j) = \frac{\left( \hat{\psi}_y (r^2_i, r^1_j) - \hat{\psi}_z (r^2_i, r^1_j) \right)^2}{\sqrt{\text{Var} \left( \hat{\psi}_y (r^2_i, r^1_j) - \hat{\psi}_z (r^2_i, r^1_j) \right)}},
\]

(52)

where

\[
\text{Var} \left[ \hat{\psi}_y (r^m_i, r^m_j) - \hat{\psi}_z (r^m_i, r^m_j) \right] = \text{Var} \left[ \hat{\psi}_y (r^m_i, r^m_j) \right] + \text{Var} \left[ \hat{\psi}_z (r^m_i, r^m_j) \right] - 2\text{Cov} \left[ \hat{\psi}_y (r^m_i, r^m_j), \hat{\psi}_z (r^m_i, r^m_j) \right],
\]

with

\[
\text{Var} \left[ \hat{\psi}_y (r^1_i, r^2_j) \right] = \frac{4\nu^2 y + 2}{T_y},
\]

\[
\text{Var} \left[ \hat{\psi}_z (r^1_i, r^2_j) \right] = \frac{4\rho^2 z + 2}{T_z},
\]

\[
\text{Cov} \left[ \hat{\psi}_y (r^1_i, r^2_j), \hat{\psi}_z (r^1_i, r^2_j) \right] = \frac{T_y}{T_z} \text{Var} \left[ \hat{\psi}_y (r^1_i, r^2_j) \right] = \frac{4\nu^2 y + 2}{T_z},
\]

where \( \hat{\psi}_y \) and \( \hat{\psi}_z \) are defined as above.

Under the null hypothesis of no contagion, all coskewness tests of contagion are
asymptotically distributed as \( \chi^2_1 \).

6 Application to the Hong Kong Crisis

6.1 Background

A common feature of many of the countries affected by the Asian financial crisis in 1997-
98, was falling real estate prices just prior to and during the crisis, with large exposures
to banks and non-bank financial intermediaries. The subsequent falls in currency and
equity markets around the region as well as the sustained recessions that most countries experienced raises the proposition that real estate markets had an important role in fuelling the crisis (Quigley (2001) and Kim and Lee (2002)). On the other hand, equity market volatility during the crisis is likely to spillover into real estate markets as the wealth effects of liquidating holdings and the taking of capital losses constrains the ability of investors to engage in other forms of investment, including those in real estate. Tests of the strength of these linkages have been undertaken by Kallberg, Liu and Pasquariello (2002); Lu and So (2005) and Knight, Lizieri and Satchell (2005). The work by Knight, Lizieri and Satchell (2005) is perhaps the closest to the present approach as they look at the relationship between higher order moments in returns by focussing on the tails of the joint distribution using copulas. The work by Kallberg, Liu and Pasquariello (2002) is also related to the present approach as they perform Granger causality tests on both the levels and volatilities of asset returns, where the last test is related to testing for cokurtosis.

The tests for contagion developed in Section 5 are applied to identify potential contagious linkages in securitised real estate and equity markets during the Asian financial crisis, after the Hong Kong shock of October 1997. The countries considered include Hong Kong, Japan and Singapore, with the United States used to control for global common factors. Although Japan and Singapore are not directly considered to be Asian crisis countries they do appear to be affected by the crisis, and their securitised real estate markets are deep compared to other crisis countries such as Thailand and Malaysia.

In the empirical analysis, an extensive set of contagion channels are investigated by testing the following linkages:

1. Contagion between real estate markets of different countries.

2. Contagion between equity markets of different countries.
3. Contagion between real estate and equity markets within countries.

4. Contagion across real estate and equity markets of different countries.

The first set of linkages are investigated by Wilson and Zurbruegg (2004) and Bond, Dungey and Fry (2006). The second set of linkages represents the main focus of most empirical models of contagion conducted on the Asian crisis (Baig and Goldfajn (1999), Forbes and Rigobon (2002), Caporale, Cipollini and Spagnolo (2003), Bekaert, Harvey and Ng (2005) and Baur and Schulze (2005), amongst others). The third set of linkages are studied by Lu and So (2005) who test for Granger causality between real estate and equity markets. The fourth set of linkages are yet to be examined, although there is other work which focusses on cross linkages amongst other asset classes (Granger, Huang and Yang (2000), Fang and Miller (2002), Khalid and Kawai (2003) and Dungey and Martin (2006)). The majority of the empirical work is based on correlation tests of contagion which do not look at higher order moment interactions such as coskewness.

6.2 Data Definitions and Description

The data consist of daily percentage returns of equity and real estate markets in Hong Kong (HK), Japan (JP), Singapore (SG) and the United States (US), all denominated in US dollars. The returns are computed as the difference of the natural logarithms of daily price indices, multiplied by 100. The real estate market indices as well as the choice of the crisis period and noncrisis period dates are taken from Bond, Dungey and Fry (2006). The real estate market indices are based on the share prices of large companies who derive at least 60 percent of their income from property related activities and are listed on the relevant stock exchange. All returns are plotted in Figure 2.

---

11The real estate market indices are from Datastream. The mnemonics are: RUHK (Hong Kong), RUJP (Japan), RUSI (Singapore) and RUUS (United States). The equity market indices are from Bloomberg. The mnemonics are: HSI (Hong Kong), TPX (Japan), SESALL (Singapore) and SPX (United States).

12As real estate returns are based on indices of listed companies, they do not include unsecuritised assets and hence do not necessarily capture the characteristics of all real estate markets (Glascock,
The total sample period of returns begins January 1, 1996 and ends June 30, 1998, a total of $T_z = 652$ observations. The noncrisis period is January 1, 1996 to October 17, 1997, a total of $T_x = 470$ observations, and the crisis period is October 20, 1997 to June 30, 1998, a total of $T_y = 182$ observations.

Table 1 presents real estate and equity market returns during the first two weeks of the Hong Kong crisis, beginning October 20th, 1997. The Hong Kong market records the greatest daily falls during this period, with returns on real estate falling by nearly 13 percent and equity by nearly 16 percent, both on October 28th. The real estate market in Singapore is particularly affected during the crisis with consecutive falls in the first week ranging from 2.57% to 8.22%, with the largest fall of 10.90% occurring in the second week. The United States experiences large movements in both markets on October 27th in the second week of the crisis where its real estate returns fall by 6.55% and equity returns fall by 3.63%.

### 6.3 Descriptive Statistics

In conducting tests of contagion it is important to extract any common factors in the data. The approach adopted is to regress the returns in Hong Kong, Japan and Singapore on the returns in the equivalent US asset market, and use the residuals from these regressions as the adjusted returns data. This approach is motivated by the filtering methods for extracting common factors adopted by Forbes and Rigobon (2002). All empirical tests presented below are conducted on the adjusted returns.

Some descriptive statistics of the unadjusted returns are given in Table 2. The statistics are reported for the noncrisis and crisis periods, as well as for the total period. Table 2 gives the variances (diagonal), covariances (lower triangle) and the correlations (upper triangle) for the three sample periods. A comparison of the variances shows that all asset markets experience substantial increases in volatility during the crisis period,

---

Lu and So (2000)). However, this is not regarded as being too restrictive for the present empirical application, especially in the case of Hong Kong whose real estate market is dominated by a small number of conglomerates which are listed on the stock exchange (Fung and Forrest (2002)).
with the absolute increase in volatility tending to be larger in real estate markets than equity markets. All correlations increase as well, with the exception of the correlation between the Hong Kong and Japanese equity markets, which remains constant at around 0.35 for the three periods.

Tests of coskewness between pairs of asset markets are given in Table 3 for the three sample periods using (33). The off-diagonal terms represent tests of coskewness, while the diagonal terms represent tests of own coskewness, namely skewness. Under the null hypothesis, the statistic is distributed asymptotically as $\chi^2_1$, with a corresponding 5% critical value of 3.84. These results show very strong evidence of coskewness amongst all real estate markets during the crisis period. There is also strong evidence of coskewness between the volatility of real estate returns and the level of equity returns in the crisis period. In contrast, there is no significant evidence during this period of coskewness between the level of real estate returns and the volatility of equity returns, or even amongst equity returns. The first result is consistent with the importance of the risk-return tradeoff in real estate markets, while the second result is consistent with the pricing of spillover risk from real estate to equity markets, as discussed in Section 2.

6.4 Evidence of Contagion

Tables 4 to 6 give the results of the correlation and coskewness contagion tests, with the results summarised in Table 7. Under the null hypothesis of no contagion, all test statistics are distributed asymptotically as $\chi^2_1$, where the 5% critical value is 3.84.

6.4.1 Correlation Contagion Tests

Table 4 presents the correlation tests of contagion using $FR_1$, $FR_2$ and $FR_3$. The $FR_2$ results in Panel B of Table 4, shows strong evidence of contagion within countries from real estate markets to equity markets. There is some evidence of contagion running in the opposite direction from equity to real estate for Hong Kong. There is also evidence of contagion occurring across countries: from equity markets in Hong Kong
Figure 2: Daily returns on real estate and equity markets for Hong Kong (HK), Japan (JP), Singapore (SG) and the United States (US): January 1 1996 to June 30 1998.
Table 1:
Returns (daily percentage) in real estate and equity markets for the period 20 October 1997 to 31 October 1997.

<table>
<thead>
<tr>
<th>Date</th>
<th>Real Estate</th>
<th></th>
<th></th>
<th></th>
<th>Equity</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
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<tr>
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<td>JP</td>
<td>SG</td>
<td>US</td>
<td></td>
<td>HK</td>
<td>JP</td>
<td>SG</td>
</tr>
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<td>-5.83</td>
<td>0.47</td>
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<td>-1.29</td>
<td>-2.26</td>
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</tr>
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<td>-3.51</td>
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<td>-1.00</td>
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</tr>
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<td>-4.35</td>
<td>3.74</td>
<td>-1.08</td>
<td>1.65</td>
</tr>
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Table 2:
Variances (diagonal), covariances (lower triangle) and correlations (upper triangle) of financial returns for selected markets and sample periods.

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Table 3:
Tests of contemporaneous coskewness (off-diagonal) and skewness (diagonal) on adjusted returns using (33). Test statistics are asymptotically distributed as chi-squared with one degree of freedom. The 5% critical value is 3.84.

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<th>Total Period</th>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

30
and Singapore to the equity market in Japan; and from the Singaporean equity market to the real estate market in Hong Kong. Interestingly, the correlation test finds no evidence of contagion amongst any of the real estate markets.

Apart from the bidirectional contagion linkages between equity and real estate markets in Hong Kong identified by $FR_1$ in Panel A of Table 4, this test does not identify any additional linkages. This result is consistent with the discussion of the properties of this statistic given in Section 5, and reflects that the test is oversized thereby biased towards failing to identify contagion when it exists.\footnote{Not surprisingly, Wilson and Zurbruegg (2004) also find no evidence of contagion amongst real estate markets using the Forbes and Rigobon test.} A comparison of the results of the $FR_3$ test in Panel C and the $FR_2$ test in Panel B, shows that the two tests yield similar qualitative results in about two-thirds of the cases. However, as the correlations in Table 2 are statistically significant for all pairs of asset returns in all three sample periods, this constitutes a violation of the independence assumption underlying $FR_3$, which may lead to false inferences. This suggests that $FR_2$ yields the more reliable inferences of the two tests.\footnote{See Appendix D for the independence tests.}

6.4.2 Contemporary Coskewness Contagion Tests

The results of the contemporaneous coskewness tests are presented in Table 5. Given the discussion of the $FR_1$ and $FR_3$ test results above, just the $CS_2$ test results are reported for the same reason, with the results for $CS_1$ and $CS_3$ presented in Appendix D.

The $CS_2$ test reveals strong evidence of contagion within real estate markets, but not equity markets. This is true for either version of the test where the source is the level of returns (Panel A), or the square of returns (Panel B). These results are in sharp contrast to the $FR_2$ results in Panel B of Table 4, where there is no significant evidence of contagion in real estate markets, while there is some evidence of contagion amongst equity markets. Panel B of Table 5 shows significant and widespread cross-asset contagious linkages from real estate markets to equity markets. These linkages occur both
within countries and across countries, and are in contrast with the correlation results in Panel B of Table 4 which also reveal linkages within countries, but not across countries.

One important qualitative difference in the coskewness test results in Panels A and B of Table 5, is that there are stronger cross asset market linkages from real estate to equity with the source country measured in squares (Panel B), while there are stronger linkages from equity to real estate with the source measured in terms of the level of returns (Panel A). The former linkage is consistent with the mean-variance tradeoff structure of the MGARCH model discussed in Section 2. The latter is consistent with allowing for an asymmetric levels effect arising from lagged equity returns in the conditional variance of real estate returns.

6.4.3 Dynamic Coskewness Contagion Tests

Coskewness tests where the source country’s returns are lagged by one period are presented in Table 6. In particular, the tests are computed as

\[
CS_2(i \rightarrow j; r_{i,t-1}^1, r_{j,t}^2) = \left( \frac{\hat{\psi}_y (r_{i,t-1}^1, r_{j,t}^2) - \hat{\psi}_x (r_{i,t-1}^1, r_{j,t}^2)}{\sqrt{4\hat{\nu}_{yi xi} (r_{i,t-1}, r_j) + 2 \hat{\rho}_z^2 (r_{i,t-1}, r_j) + 2}} \right)^2
\]  
\text{(53)}

\[
CS_2(i \rightarrow j; r_{i,t-1}^2, r_{j,t}^1) = \left( \frac{\hat{\psi}_y (r_{i,t-1}^2, r_{j,t}^1) - \hat{\psi}_x (r_{i,t-1}^2, r_{j,t}^1)}{\sqrt{4\hat{\nu}_{yi xi} (r_{i,t-1}, r_j) + 2 \hat{\rho}_z^2 (r_{i,t-1}, r_j) + 2}} \right)^2
\]  
\text{(54)}

As with the contemporaneous coskewness tests, just the results of the $CS_2$ version of the test are reported, with the results for the dynamic versions of $CS_1$ and $CS_3$ given in Appendix D.

The main difference between the contemporaneous coskewness results in Table 5 and the dynamic coskewness results in Table 6, is the prevalence of coskewness amongst equity markets in the dynamic results which did not exist in the contemporaneous results. As with the contemporaneous results, the dynamic coskewness tests also identify
important contagious linkages amongst real estate markets, and between real estate and equity markets within and across countries.

7 Conclusions

This paper has introduced a new class of tests of contagion based on testing for changes in coskewness during financial crises. The tests extended the existing correlation based class of contagion tests by identifying changes in contagion through the interaction of the level and volatility of returns. This was shown to be an important and natural link to test for contagion as it captured the portfolio effects of changes to the risk-return trade-offs in pricing risk both within asset markets, and the spillover risk across different classes of asset markets.

The framework was applied to identifying contagion in real estate and equity markets both within and across national borders during the aftermath of the Hong Kong equity market crash in October, 1997. The empirical results provided strong refutation of the Forbes and Rigobon (2002) empirical results as there was widespread evidence of contagion running across different asset markets within a country, across the same class of asset markets in different countries, and across different asset markets in different countries. These results not only arose from testing for additional channels of contagion which were ignored by correlation tests, but also due to an adjustment of the standard errors in the correlation test that corrected for potential size distortions.
Table 4:
Correlation tests of contagion on adjusted returns based on (37), (40) and (46). The 5% critical value is 3.84.

Panel A: $FR_1 (i \rightarrow j)$ test

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<tr>
<th>Source ($i$)</th>
<th>Real Estate</th>
<th>Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HK</td>
<td>JP</td>
</tr>
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<td><strong>Real Estate</strong></td>
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<td>0.04 0.67</td>
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<tr>
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</tr>
<tr>
<td></td>
<td>SG 3.21 0.06 n.a.</td>
<td>2.38 0.00 2.92</td>
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</table>

Panel B: $FR_2 (i \rightarrow j)$ test

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<th>Equity</th>
</tr>
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</tr>
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<td>SG 0.83 0.22 n.a.</td>
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Panel C: $FR_3 (i \rightarrow j)$ test

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</tr>
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<td>SG 5.65 0.10 n.a.</td>
<td>4.20 0.00 5.14</td>
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34
Table 5:
Contemporaneous coskewness tests of contagion on adjusted returns, using (49) and (50). The 5% critical value is 3.84.

<table>
<thead>
<tr>
<th>Panel A: $CS_2 (i \rightarrow j; r_i^1, r_j^2)$ test</th>
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<th>Equity</th>
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<table>
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Table 6:
Dynamic coskewness tests of contagion on adjusted returns, using (53) and (54). The 5% critical value is 3.84.

Panel A: $CS_2 (i \rightarrow j; r_{i,t-1}^1, r_{j,t}^2)$ test

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Panel B: $CS_2 (i \rightarrow j; r_{i,t-1}^2, r_{j,t}^1)$ test

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Table 7:
Summary of the empirical results of the $FR_2$ and $CS_2$ contagion tests.

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<th>Recipient ($j$)</th>
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<th>Dynamic</th>
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<td>$CS_2 (y_{i,t}^1)$ (49)</td>
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<tr>
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<td>SG</td>
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</tr>
</tbody>
</table>

37
\[ M(t_1, t_2) = \exp \left[ \frac{1}{2} \left( 2 \mu_1 t_1 + 2 \mu_2 t_2 + \sigma_1^2 t_1^2 + 2 \rho \sigma_1 \sigma_2 t_1 t_2 + \sigma_2^2 t_2^2 \right) \right] \]

\[ = 1 + \frac{1}{2^1 \times 2!} \left[ 2 \mu_1 t_1 + 2 \mu_2 t_2 + \sigma_1^2 t_1^2 + 2 \rho \sigma_1 \sigma_2 t_1 t_2 + \sigma_2^2 t_2^2 \right] \]

\[ + \frac{1}{2^2 \times 2^2} \left[ 8 \mu_1^2 t_1^2 \mu_1 + 8 \rho \sigma_1^2 \sigma_2^2 t_1 \sigma_2 + 8 \rho^2 \sigma_1^2 \sigma_2^2 t_1 \sigma_2 + 4 \mu_1^2 \sigma_2^2 + 4 \mu_2^2 \sigma_2^2 \right] \]

\[ + \frac{1}{2^3 \times 3!} \left[ 16 \mu_1^2 \mu_2^2 + 8 \mu_1^2 \sigma_1^2 t_1^2 \sigma_1 + 8 \mu_1^2 \sigma_1^2 t_2^2 \sigma_1 + 8 \mu_1^2 \sigma_2^2 t_1^2 \sigma_2 + 8 \mu_1^2 \sigma_2^2 t_2^2 \sigma_2 + 8 \rho \sigma_1 \sigma_2 t_1^2 \sigma_1 + 8 \rho \sigma_1 \sigma_2 t_2^2 \sigma_2 + 8 \rho \sigma_1 \sigma_2 t_1^2 \sigma_2 + 8 \rho \sigma_1 \sigma_2 t_2^2 \sigma_1 \right] \]

\[ + \frac{1}{2^4 \times 4!} \left[ +24 \mu_1^{\frac{3}{2}} \mu_2^{\frac{3}{2}} + 24 \mu_1^{\frac{3}{2}} t_1^2 \sigma_1^2 + 24 \mu_1^{\frac{3}{2}} t_2^2 \sigma_1^2 + 24 \mu_1^{\frac{3}{2}} t_1^2 \sigma_2^2 + 24 \mu_1^{\frac{3}{2}} t_2^2 \sigma_2^2 \right] \]

\[ + \frac{1}{2^5 \times 5!} \left[ +12 \mu_1^2 \mu_2^2 + 12 \mu_1^2 \sigma_1^2 t_1^2 \sigma_1 + 12 \mu_1^2 \sigma_1^2 t_2^2 \sigma_1 + 12 \mu_1^2 \sigma_2^2 t_1^2 \sigma_2 + 12 \mu_1^2 \sigma_2^2 t_2^2 \sigma_2 + 12 \mu_1^2 t_1^2 \sigma_1^2 \sigma_2 + 12 \mu_1^2 t_2^2 \sigma_1^2 \sigma_2 \right] \]

\[ + \frac{1}{2^6 \times 6!} \left[ +192 \mu_1^{\frac{5}{2}} \mu_2^{\frac{5}{2}} + 192 \mu_1^{\frac{5}{2}} t_1^2 \sigma_1^2 \sigma_2 + 192 \mu_1^{\frac{5}{2}} t_2^2 \sigma_1^2 \sigma_2 + 192 \mu_1^{\frac{5}{2}} t_1^2 \sigma_2^2 \sigma_1 + 192 \mu_1^{\frac{5}{2}} t_2^2 \sigma_2^2 \sigma_1 + 192 \mu_1^{\frac{5}{2}} t_1^2 \sigma_1^2 \sigma_2 + 192 \mu_1^{\frac{5}{2}} t_2^2 \sigma_2^2 \sigma_2 \right] \]

\[ + \ldots \]

As the moment generating function of a bivariate normal distribution is used in the derivations, it is presented here for completeness (Kendall and Stuart (1969), p.82)
This expression is used to extract the following moments

\[
\begin{align*}
E[r_{1,t}] &= \sigma_1^2 + \mu_1^2, \\
E[r_{2,t}] &= \sigma_2^2 + \mu_2^2, \\
E[r_{1,t}^2] &= 3\sigma_1^4 + \mu_1^4 + 6\sigma_1^2\mu_1^2, \\
E[r_{2,t}^2] &= 3\sigma_2^4 + \mu_2^4 + 6\sigma_2^2\mu_2^2, \\
E[r_{1,t}r_{2,t}] &= \mu_1\mu_2 + \rho\sigma_1\sigma_2, \\
E[r_{1,t}r_{2,t}^2] &= 4\rho\sigma_1\sigma_2\mu_1\mu_2 + \sigma_1^2\sigma_2^2 + \sigma_1^2\mu_2^2 + \sigma_2^2\mu_1^2 + 2\rho^2\sigma_1^2\sigma_2^2, \\
E[r_{1,t}r_{2,t}^3] &= 9\rho\sigma_1^3\sigma_2^3 + 9\rho\sigma_1\sigma_2^3\mu_1^2 + 9\rho\sigma_1^3\sigma_2\mu_2^2 + 9\sigma_1^2\sigma_2^2\mu_1\mu_2 + 6\rho^3\sigma_1^3\sigma_2^3 + 18\rho^2\sigma_1^2\sigma_2^2\mu_1\mu_2, \\
E[r_{1,t}r_{2,t}^4] &= 9\sigma_1^4\sigma_2^4 + 72\rho^2\sigma_1^3\sigma_2^3 + 24\rho^4\sigma_1^4\sigma_2^4, \\
E[r_{1,t}r_{2,t}^6] &= E[r_{1,t}r_{2,t}^3] = 0, \\
E[r_{1,t}r_{2,t}^7] &= \mu_1\mu_2^3 + 3\rho\sigma_1\sigma_2^3 + 3\sigma_2^2\mu_1\mu_2 + 3\rho\sigma_1\sigma_2\mu_2^2, \\
E[r_{1,t}r_{2,t}^8] &= \mu_2\mu_1^3 + 3\rho\sigma_1^3\sigma_2 + 3\sigma_2\mu_1\mu_2 + 3\rho\sigma_1\sigma_2\mu_1^2, \\
E[r_{1,t}r_{2,t}^9] &= E[r_{1,t}r_{2,t}^6] = 0, \\
E[r_{1,t}r_{2,t}^{10}] &= 15\rho\sigma_1^5\sigma_2^5 + 15\sigma_2^4\mu_1\mu_2 + 30\rho\sigma_1^3\sigma_2^3\mu_2^2, \\
E[r_{1,t}r_{2,t}^{11}] &= 15\rho\sigma_1^5\sigma_2 + 15\sigma_2^4\mu_1\mu_2 + 30\rho\sigma_1^3\sigma_2\mu_2^2, \\
E[r_{1,t}r_{2,t}^{12}] &= E[r_{1,t}r_{2,t}^9] = 0, \\
E[r_{1,t}r_{2,t}^{13}] &= 24\rho\sigma_1^3\sigma_2\mu_1\mu_2 + 3\sigma_1^2\sigma_2^4 + 3\sigma_1^4\mu_1^2 + 12\rho^2\sigma_1^2\sigma_2^4 + 6\sigma_1^2\sigma_2^2\mu_2^2 + 12\rho^2\sigma_1^2\sigma_2^2\mu_2^2, \\
E[r_{1,t}r_{2,t}^{14}] &= 24\rho\sigma_1^3\sigma_2\mu_1\mu_2 + 3\sigma_1^4\sigma_2^2 + 3\sigma_1^4\mu_2^2 + 12\rho^2\sigma_1^4\sigma_2^2 + 6\sigma_1^2\sigma_2^2\mu_1^2 + 12\rho^2\sigma_1^2\sigma_2^2\mu_1^2, \\
E[r_{1,t}r_{2,t}^{15}] &= E[r_{1,t}r_{2,t}^{12}] = 0, \\
E[r_{1,t}r_{2,t}^{16}] &= 15\sigma_1^6\sigma_2^6 + 90\rho^2\sigma_1^2\sigma_2^6, \\
E[r_{1,t}r_{2,t}^{17}] &= 15\sigma_2^6\sigma_2^2 + 90\rho^2\sigma_1^2\sigma_2^6.
\end{align*}
\]
B Information Matrix Derivations

The following results are used to derive the information matrix of the various test statistics below. The expectations use the properties of the moment generating function of a bivariate normal distribution given in Appendix A.

B.1 Bivariate Normal

For the bivariate normal distribution

$$h_t = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right].$$

The pertinent first derivatives are

$$\frac{\partial h_t}{\partial \mu_1} = \frac{1}{\sigma_1} \frac{1}{1 - \rho^2} \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) - \rho \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right]$$

$$\frac{\partial h_t}{\partial \mu_2} = \frac{1}{\sigma_2} \frac{1}{1 - \rho^2} \left[ \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \right]$$

$$\frac{\partial h_t}{\partial \sigma_1^2} = \frac{1}{2 \sigma_1^2} \frac{1}{1 - \rho^2} \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right]$$

$$\frac{\partial h_t}{\partial \sigma_2^2} = \frac{1}{2 \sigma_2^2} \frac{1}{1 - \rho^2} \left[ \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right]$$

$$\frac{\partial h_t}{\partial \rho} = \left( \frac{1}{1 - \rho^2} \right)^2 \left[ (1 + \rho^2) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right.\left. - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right].$$
Taking expectations of these derivatives gives

\[
E \left[ \frac{\partial h_1}{\partial \mu_1} \right] = E \left[ \frac{1}{\sigma_1} \frac{1}{1 - \rho^2} \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) - \rho \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right] = 0
\]

\[
E \left[ \frac{\partial h_1}{\partial \mu_2} \right] = E \left[ \frac{1}{\sigma_2} \frac{1}{1 - \rho^2} \left( \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \right) \right] = 0
\]

\[
E \left[ \frac{\partial h_1}{\partial \sigma_1} \right] = E \left[ \frac{1}{2} \frac{1}{\sigma_1} \left( \frac{1}{1 - \rho^2} \right) \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right] = \frac{1}{2\sigma_1^2}
\]

\[
E \left[ \frac{\partial h_1}{\partial \sigma_2} \right] = E \left[ \frac{1}{2} \frac{1}{\sigma_2} \left( \frac{1}{1 - \rho^2} \right) \left( \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right] = \frac{1}{2\sigma_2^2}
\]

\[
E \left[ \frac{\partial h_1}{\partial \rho} \right] = E \left[ \left( \frac{1}{1 - \rho^2} \right)^2 \left( 1 + \rho^2 \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right. \\
- \rho \left( \frac{1}{1 - \rho^2} \right)^2 \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right) \left] = \frac{\rho}{1 - \rho^2}
\]

where the following results are used

\[
\mu_i = E \left[ r_{i,t} \right]; E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^2 \right] = 1; \rho = E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right) \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right) \right], i \neq j.
\]

In addition

\[
E \left[ \left( \frac{\partial h}{\partial \mu_1} \right)^2 \right] = E \left[ \left( \frac{1}{\sigma_1} \frac{1}{1 - \rho^2} \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) - \rho \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right)^2 \right] = \frac{1}{\sigma_1^2} \frac{1}{1 - \rho^2}
\]
\[
E \left[ \left( \frac{\partial h}{\partial \mu_1} \right) \left( \frac{\partial h}{\partial \mu_2} \right) \right] = E \left[ \left( \frac{1}{\sigma_1} \frac{1}{1 - \rho^2} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) - \rho \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right]
\left[ \frac{1}{\sigma_2} \frac{1}{1 - \rho^2} \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \right] \\
= -\frac{1}{\sigma_1 \sigma_2} \rho \left( \frac{1}{1 - \rho^2} \right)
\]

\[
E \left[ \left( \frac{\partial h}{\partial \mu_1} \right) \left( \frac{\partial h}{\partial \sigma_1^2} \right) \right] = E \left[ \left( \frac{1}{\sigma_1} \frac{1}{1 - \rho^2} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) - \rho \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right]
\left[ \frac{1}{2 \sigma_1^2} \frac{1}{1 - \rho^2} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \right.
\left. - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] \\
= 0
\]

\[
E \left[ \left( \frac{\partial h}{\partial \mu_1} \right) \left( \frac{\partial h}{\partial \sigma_2^2} \right) \right] = E \left[ \left( \frac{1}{\sigma_1} \frac{1}{1 - \rho^2} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) - \rho \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right]
\left[ \frac{1}{2 \sigma_2^2} \frac{1}{1 - \rho^2} \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right.
\left. - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] \\
= 0
\]

\[
E \left[ \left( \frac{\partial h}{\partial \mu_1} \right) \left( \frac{\partial h}{\partial \rho} \right) \right] = E \left[ \left( \frac{1}{\sigma_1} \frac{1}{1 - \rho^2} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) - \rho \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right]
\left[ \left( \frac{1}{1 - \rho^2} \right)^2 \left( 1 + \rho^2 \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right.$
\left. - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right) \right] \\
= 0
\]

\[
E \left[ \frac{\partial h}{\partial \mu_2} \right]^2 = E \left[ \left( \frac{1}{\sigma_2} \frac{1}{1 - \rho^2} \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \right)^2 \right]
\left( \frac{1}{\sigma_2} \right)^2 \left( \frac{1}{1 - \rho^2} \right)
\]
\[ E \left[ \left( \frac{\partial h}{\partial \mu_2} \right) \left( \frac{\partial h}{\partial (\sigma_1^2)} \right) \right] = E \left[ \frac{1}{\sigma_2} \left( 1 - \rho^2 \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \right) \right] \]

\[ = 0 \]

\[ E \left[ \left( \frac{\partial h}{\partial \mu_2} \right) \left( \frac{\partial h}{\partial (\sigma_2^2)} \right) \right] = E \left[ \frac{1}{\sigma_2} \left( 1 - \rho^2 \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \right) \right] \]

\[ = 0 \]

\[ E \left[ \left( \frac{\partial h}{\partial \mu_2} \right) \left( \frac{\partial h}{\partial \rho} \right) \right] = E \left[ \frac{1}{\sigma_2} \left( 1 - \rho^2 \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \right) \right] \]

\[ = 0 \]

\[ E \left[ \left( \frac{\partial h}{\partial \sigma_1^2} \right)^2 \right] = E \left[ \frac{1}{4 \sigma_1^4} \left( 1 - \rho^2 \right)^2 \right] \]

\[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \]

\[ = \frac{1}{4 \sigma_1^4} \left( 1 - \rho^2 \right)^2 \left( 3 - 5\rho^2 + 2\rho^4 \right) \]
\[
E \left[ \left( \frac{\partial h}{\partial \sigma_1^2} \right) \left( \frac{\partial h}{\partial \sigma_2^2} \right) \right] = E \left[ \left( \frac{1}{2} \frac{1}{\sigma_1^2} \left( \frac{1}{1-\rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \right. \right.
\left. \left. - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] \right.
\left. \left[ \frac{1}{2} \frac{1}{\sigma_2^2} \left( \frac{1}{1-\rho^2} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right. \right.
\left. \left. - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] \right] \right.
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\[
E \left[ \left( \frac{\partial h}{\partial \sigma_2^2} \right) \left( \frac{\partial h}{\partial \rho} \right) \right] = E \left[ \left( \frac{1}{2} \frac{1}{\sigma_2^2} \frac{1}{1 - \rho^2} \right) \left( \left( \frac{r - \mu_2}{\sigma_2} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right] \\
\left[ \left( \frac{1}{1 - \rho^2} \right)^2 \left( 1 + \rho^2 \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right] \\
- \rho \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right) \right] \right] \\
= - \frac{1}{\sigma_2^2} \rho \left( \frac{1}{1 - \rho^2} \right)
\]

\[
E \left[ \left( \frac{\partial h}{\partial \rho} \right)^2 \right] = E \left[ \left( \frac{1}{1 - \rho^2} \right)^2 \left( 1 + \rho^2 \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right) \right] \\
- \rho \left( \left( \frac{1}{1 - \rho^2} \right)^2 \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right) \right)^2 \right] \\
= \left( \frac{1}{1 - \rho^2} \right)^4 (2 \rho^6 - 3 \rho^4 + 1),
\]

where the following results are used from Appendix A

\[
E \left[ \left( \frac{r_i - \mu_i}{\sigma_i} \right)^3 \right] = E \left[ \left( \frac{r_i - \mu_i}{\sigma_i} \right)^4 \left( \frac{r_j - \mu_j}{\sigma_j} \right)^2 \right] = 0
\]

\[
E \left[ \left( \frac{r_i - \mu_i}{\sigma_i} \right)^4 \right] = 3
\]

\[
E \left[ \left( \frac{r_i - \mu_i}{\sigma_i} \right)^2 \left( \frac{r_j - \mu_j}{\sigma_j} \right)^2 \right] = 1 + 2 \rho^2
\]

\[
E \left[ \left( \frac{r_i - \mu_i}{\sigma_i} \right)^4 \left( \frac{r_j - \mu_j}{\sigma_j} \right)^3 \right] = 3 \rho, i \neq j.
\]

The elements of the information matrix at observation \( t \), are then

\[
I_{1,1,t} = E \left[ \left( \frac{\partial h}{\partial \mu_1} \right)^2 \right] - E \left[ \frac{\partial h}{\partial \mu_1} \right] E \left[ \frac{\partial h}{\partial \mu_1} \right] \\
= \frac{1}{\sigma_1^2} \frac{1}{1 - \rho^2} \left( 0 \right) \left( 0 \right) \\
= \frac{1}{\sigma_1^2} \frac{1}{1 - \rho^2}
\]

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\[ I_{1,2,t} = E \left[ \left( \frac{\partial h}{\partial \mu_1} \right) \left( \frac{\partial h}{\partial \mu_2} \right) \right] - E \left[ \frac{\partial h}{\partial \mu_1} \right] E \left[ \frac{\partial h}{\partial \mu_2} \right] \]
\[ = - \frac{1}{\sigma_1 \sigma_2} \rho \left( \frac{1}{1 - \rho^2} \right) - (0) (0) \]
\[ = - \frac{1}{\sigma_1 \sigma_2} \rho \left( \frac{1}{1 - \rho^2} \right) \]

\[ I_{1,3,t} = E \left[ \left( \frac{\partial h}{\partial \mu_1} \right) \left( \frac{\partial h}{\partial (\sigma_1^2)} \right) \right] - E \left[ \frac{\partial h}{\partial \mu_1} \right] E \left[ \frac{\partial h}{\partial (\sigma_1^2)} \right] \]
\[ = 0 - (0) \left( \frac{1}{2\sigma_1^2} \right) \]
\[ = 0 \]

\[ I_{1,4,t} = E \left[ \left( \frac{\partial h}{\partial \mu_1} \right) \left( \frac{\partial h}{\partial \sigma_2} \right) \right] - E \left[ \frac{\partial h}{\partial \mu_1} \right] E \left[ \frac{\partial h}{\partial \sigma_2} \right] \]
\[ = 0 - (0) \left( \frac{1}{2\sigma_2^2} \right) \]
\[ = 0 \]

\[ I_{1,5,t} = E \left[ \left( \frac{\partial h}{\partial \mu_1} \right) \left( \frac{\partial h}{\partial \rho} \right) \right] - E \left[ \frac{\partial h}{\partial \mu_1} \right] E \left[ \frac{\partial h}{\partial \rho} \right] \]
\[ = 0 - (0) \left( - \left( \frac{\rho}{1 - \rho^2} \right) \right) \]
\[ = 0 \]

\[ I_{2,2,t} = E \left[ \left( \frac{\partial h}{\partial \mu_2} \right)^2 \right] - E \left[ \frac{\partial h}{\partial \mu_2} \right] E \left[ \frac{\partial h}{\partial \mu_2} \right] \]
\[ = \left( \frac{1}{\sigma_2} \right)^2 \left( \frac{1}{1 - \rho^2} \right) - (0) (0) \]
\[ = \left( \frac{1}{\sigma_2} \right)^2 \left( \frac{1}{1 - \rho^2} \right) \]

\[ I_{2,3,t} = E \left[ \left( \frac{\partial h}{\partial \mu_2} \right) \left( \frac{\partial h}{\partial (\sigma_1^2)} \right) \right] - E \left[ \frac{\partial h}{\partial \mu_2} \right] E \left[ \frac{\partial h}{\partial (\sigma_1^2)} \right] \]
\[ = 0 - (0) \left( \frac{1}{2\sigma_1^2} \right) \]
\[ = 0 \]
\[ I_{2,4,t} = E \left[ \left( \frac{\partial h}{\partial \mu_2} \right) \left( \frac{\partial h}{\partial \sigma_2^2} \right) \right] - E \left[ \frac{\partial h}{\partial \mu_2} \right] E \left[ \frac{\partial h}{\partial \sigma_2^2} \right] \\
= 0 - (0) \left( \frac{1}{2 \sigma_2^2} \right) \\
= 0 \]

\[ I_{2,5,t} = E \left[ \left( \frac{\partial h}{\partial \mu_2} \right) \left( \frac{\partial h}{\partial \rho} \right) \right] - E \left[ \frac{\partial h}{\partial \mu_2} \right] E \left[ \frac{\partial h}{\partial \rho} \right] \\
= 0 - (0) \left( - \left( \frac{\rho}{1 - \rho^2} \right) \right) \\
= 0 \]

\[ I_{3,3,t} = E \left[ \left( \frac{\partial h}{\partial \sigma_1^2} \right)^2 \right] - E \left[ \frac{\partial h}{\partial \sigma_1^2} \right] E \left[ \frac{\partial h}{\partial \sigma_1^2} \right] \\
= \frac{1}{4 \sigma_1^4} \left( \frac{1}{1 - \rho^2} \right)^2 (3 - 5 \rho^2 + 2 \rho^4) - \left( \frac{1}{2 \sigma_1^2} \right)^2 \\
= \frac{1}{4 \sigma_1^4} \left( \frac{1}{1 - \rho^2} \right) (2 - \rho^2) \]

\[ I_{3,4,t} = E \left[ \left( \frac{\partial h}{\partial \sigma_1^2} \right) \left( \frac{\partial h}{\partial \sigma_2^2} \right) \right] - E \left[ \frac{\partial h}{\partial \sigma_1^2} \right] E \left[ \frac{\partial h}{\partial \sigma_2^2} \right] \\
= \frac{1}{4 \sigma_1^2 \sigma_2^2} \left( \frac{1}{1 - \rho^2} \right) (1 - 2 \rho^2) - \left( \frac{1}{2 \sigma_1^2} \right) \left( \frac{1}{2 \sigma_2^2} \right) \\
= - \frac{1}{4 \sigma_1^2 \sigma_2^2} \rho^2 \left( \frac{1}{1 - \rho^2} \right) \]

\[ I_{3,5,t} = E \left[ \left( \frac{\partial h}{\partial \sigma_1^2} \right) \left( \frac{\partial h}{\partial \rho} \right) \right] - E \left[ \frac{\partial h}{\partial \sigma_1^2} \right] E \left[ \frac{\partial h}{\partial \rho} \right] \\
= - \frac{1}{\sigma_1^2} \rho \left( \frac{1}{1 - \rho^2} \right) - \left( \frac{1}{2 \sigma_1^2} \right) \left( \frac{\rho}{1 - \rho^2} \right) \\
= - \frac{1}{2 \sigma_1^2} \rho \left( \frac{1}{1 - \rho^2} \right) \]

\[ I_{4,4,t} = E \left[ \left( \frac{\partial h}{\partial \sigma_2^2} \right)^2 \right] - E \left[ \frac{\partial h}{\partial \sigma_2^2} \right] E \left[ \frac{\partial h}{\partial \sigma_2^2} \right] \\
= \frac{1}{4 \sigma_2^4} \left( \frac{1}{1 - \rho^2} \right)^2 (3 - 5 \rho^2 + 2 \rho^4) - \left( \frac{1}{2 \sigma_2^2} \right)^2 \\
= \frac{1}{4 \sigma_2^4} \left( \frac{1}{1 - \rho^2} \right) (2 - \rho^2) \]
\[ I_{4,5,t} = E \left[ \left( \frac{\partial h}{\partial \sigma_2^2} \right) \left( \frac{\partial h}{\partial \rho} \right) \right] - E \left[ \frac{\partial h}{\partial \sigma_2^2} \right] E \left[ \frac{\partial h}{\partial \rho} \right] \]
\[ = -1 \sigma_2^2 \left( \frac{1}{1 - \rho^2} \right) - \left( \frac{1}{2 \sigma_2^2} \right) \left( \frac{\rho}{1 - \rho^2} \right) \]
\[ = -1 \sigma_2^2 \left( \frac{1}{1 - \rho^2} \right) \rho \]

\[ I_{5,5,t} = E \left[ \left( \frac{\partial h}{\partial \rho} \right)^2 \right] - E \left[ \frac{\partial h}{\partial \rho} \right] E \left[ \frac{\partial h}{\partial \rho} \right] \]
\[ = \left( \frac{1}{1 - \rho^2} \right)^4 \left( 2 \rho^6 - 3 \rho^4 + 1 \right) - \left( \frac{\rho}{1 - \rho^2} \right)^2 \]
\[ = \left( \frac{1}{1 - \rho^2} \right)^2 \left( 1 + \rho^2 \right) . \]

**B.2 Coskewness**

The expression of \( h_t \) for the generalised normal distribution used to construct the coskewness test is

\[ h_t = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \]
\[ + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 . \]

The pertinent first derivatives of \( h_t \) are

\[ \frac{\partial h_t}{\partial \mu_1} = \left( \frac{1}{\sigma_1} \right) \left[ \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) - \rho \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] \]
\[ - \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \]

\[ \frac{\partial h_t}{\partial \mu_2} = \left( \frac{1}{\sigma_2} \right) \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \]
\[ - 2\phi \left( \frac{1}{\sigma_2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \]

\[ \frac{\partial h_t}{\partial \sigma_1^2} = \frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \]
\[ - \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \]
\begin{align*}
\frac{\partial h_t}{\partial \sigma_2^2} &= \left(\frac{1}{\sigma_2^2}\right) \left[\frac{1}{2} \left(\frac{1}{1 - \rho^2}\right) \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)^2 - \rho \left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right) \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)\right] \\
&\quad - \phi \left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right) \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)^2 \\
\frac{\partial h_t}{\partial \rho} &= \left(\frac{1}{1 - \rho^2}\right)^2 \left[\left(1 + \rho^2\right) \left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right) \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)\right] \\
&\quad - \rho \left(\left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right)^2 + \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)^2\right) \\
\frac{\partial h_t}{\partial \phi} &= \left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right) \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)^2.
\end{align*}

Taking the expectations of these derivative under the null hypothesis of bivariate normality \((\phi = 0)\), gives

\begin{align*}
E \left[\frac{\partial h_t}{\partial \mu_1}\right] &= E \left[\left(\frac{1}{\sigma_1}\right) \left(\left(\frac{1}{1 - \rho^2}\right) \left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right) - \rho \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)\right)\right] = 0 \\
E \left[\frac{\partial h_t}{\partial \mu_2}\right] &= E \left[\left(\frac{1}{\sigma_2}\right) \left(\frac{1}{1 - \rho^2}\right) \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right) - \rho \left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right)\right] = 0 \\
E \left[\frac{\partial h_t}{\partial \sigma_1^2}\right] &= E \left[\frac{1}{2 \sigma_1^2} \left[\left(\frac{1}{1 - \rho^2}\right) \left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right)^2 - \rho \left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right) \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)\right]\right] \\
&= \frac{1}{2 \sigma_1^2} \\
E \left[\frac{\partial h_t}{\partial \sigma_2^2}\right] &= E \left[\frac{1}{\sigma_2^2} \left(\frac{1}{2 \left(1 - \rho^2\right)} \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)^2 - \rho \left(\frac{r_{1,t} - \mu_1}{\sigma_1}\right) \left(\frac{r_{2,t} - \mu_2}{\sigma_2}\right)\right)\right] \\
&= \frac{1}{2 \sigma_2^2}
\end{align*}
\[
E \left[ \frac{\partial h_t}{\partial \rho} \right] = E \left[ \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right]
\]
\[
- \rho \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2
\]
\[
- 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)
\]
\[
= - \left( \frac{\rho}{1 - \rho^2} \right)
\]

\[
E \left[ \frac{\partial h_t}{\partial \phi} \right] = E \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] = 0
\]

where the following results are used from Appendix A

\[
\mu_i = E [r_{i,t}] ; E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^2 \right] = 1 ; E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right) \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^2 \right] = 0,
\]

and

\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right) \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right) \right] = \rho, i \neq j.
\]

Taking expectations under the null hypothesis of bivariate normality, the second order terms are
\[
E \left[ \left( \frac{\partial h}{\partial \mu_1} \right)^2 \right] = E \left[ \left( \frac{1}{\sigma_1} \right)^2 \left( \frac{1}{1 - \rho^2} \right)^2 \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \rho^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right) - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] \\
= \left( \frac{1}{\sigma_1} \right)^2 \left( \frac{1}{1 - \rho^2} \right)
\]

\[
E \left[ \left( \frac{\partial h}{\partial \mu_2} \right)^2 \right] = E \left[ \left( \frac{1}{\sigma_2} \right)^2 \left( \frac{1}{1 - \rho^2} \right)^2 \left( \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \right)^2 \right] \\
= \left( \frac{1}{\sigma_2} \right)^2 \left( \frac{1}{1 - \rho^2} \right)
\]

\[
E \left[ \left( \frac{\partial h}{\partial \sigma_1^2} \right)^2 \right] = E \left[ \frac{1}{4} \frac{1}{\sigma_1^4} \left( \frac{1}{1 - \rho^2} \right)^2 \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right)^2 \right] \\
= \frac{1}{4} \frac{1}{\sigma_1^4} \left( \frac{1}{1 - \rho^2} \right)^2 \left( 3 - 5\rho^2 + 2\rho^4 \right)
\]

\[
E \left[ \left( \frac{\partial h}{\partial \sigma_2^2} \right)^2 \right] = E \left[ \frac{1}{4} \frac{1}{\sigma_2^4} \left( \frac{1}{1 - \rho^2} \right)^2 \left( \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right)^2 \right] \\
= \frac{1}{4} \frac{1}{\sigma_2^4} \left( \frac{1}{1 - \rho^2} \right)^2 \left( 3 - 5\rho^2 + 2\rho^4 \right)
\]

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\[
E \left[ \left( \frac{\partial h}{\partial \rho} \right)^2 \right] = E \left[ (1 + \rho^2)^2 \left( \frac{1}{1 - \rho^2} \right)^4 \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right. \\
\quad \left. + \rho^2 \left( \frac{1}{1 - \rho^2} \right)^4 \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right)^2 \right] \\
\quad - 2\rho \left( 1 + \rho^2 \right) \left( \frac{1}{1 - \rho^2} \right)^4 \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \\
\quad \left. \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right) \right] \\
= \left( \frac{1}{1 - \rho^2} \right)^4 (2\rho^6 - 3\rho^4 + 1)
\]

\[
E \left[ \left( \frac{\partial h}{\partial \phi} \right)^2 \right] = E \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^4 \right] = 3 + 12\rho^2
\]

\[
E \left[ \frac{\partial h}{\partial \mu_1} \frac{\partial h}{\partial \phi} \right] = E \left[ \left( \frac{1}{\sigma_1} \right) \left( \frac{1}{1 - \rho^2} \right)^4 \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right. \\
\quad \left. - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^3 \right] \\
= \frac{1}{\sigma_1}
\]

\[
E \left[ \frac{\partial h}{\partial \mu_2} \frac{\partial h}{\partial \phi} \right] = E \left[ \left( \frac{1}{\sigma_2} \right) \left( \frac{1}{1 - \rho^2} \right)^4 \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^3 \right. \\
\quad \left. - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] \\
= \frac{2\rho}{\sigma_2}
\]

\[
E \left[ \frac{\partial h}{\partial \sigma_1^2} \frac{\partial h}{\partial \phi} \right] = E \left[ \frac{1}{2} \left( \frac{1}{\sigma_1^2} \right) \left( \frac{1}{\rho^2} \right)^4 \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^3 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right. \\
\quad \left. - \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^3 \right] \\
= 0
\]
\[
E \left[ \frac{\partial h}{\partial \sigma_2^2} \frac{\partial h}{\partial \phi} \right] = E \left[ \frac{1}{2} \left( \frac{1}{\sigma_2^2} \right) \left( \frac{1}{1 - \rho^2} \right) \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^4 \right. \right.
\]
\[
- \rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^3 \left] \right) \right] \]
\[
= 0
\]
\[
E \left[ \frac{\partial h}{\partial \rho} \frac{\partial h}{\partial \phi} \right] = E \left[ \left( \frac{1}{1 - \rho^2} \right)^2 \left( 1 + \rho^2 \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^3 \right.
\]
\[
- \rho \left( \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^3 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^4 \right) \left] \right] \]
\[
= 0,
\]

which use the following results from Appendix A
\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^{1} \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^{2} \right] = 0
\]
\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^{2} \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^{3} \right] = 0
\]
\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^{1} \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^{4} \right] = 0
\]
\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^{3} \right] = 0
\]
\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^{4} \right] = 3
\]
\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^{2} \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^{2} \right] = 1 + 2\rho^2
\]
\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^{1} \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^{3} \right] = 3\rho, i \neq j.
\]

The elements of the information matrix at observation \( t \), evaluated under the null are
\[
I_{1,1,t} = E \left[ \frac{\partial h}{\partial \mu_1} \frac{\partial h}{\partial \mu_1} \right] - E \left[ \frac{\partial h}{\partial \mu_1} \right] E \left[ \frac{\partial h}{\partial \mu_1} \right] = \left( \frac{1}{\sigma_1} \right)^2 \left( \frac{1}{1 - \rho^2} \right) - (0)^2
\]
\[
= \left( \frac{1}{\sigma_1} \right)^2 \left( \frac{1}{1 - \rho^2} \right)
\]
\[ I_{2,2,t} = E \left[ \frac{\partial h}{\partial \mu} \frac{\partial h}{\partial \mu} \right] - E \left[ \frac{\partial h}{\partial \mu} \right] E \left[ \frac{\partial h}{\partial \mu} \right] \\
= \left( \frac{1}{\sigma_2} \right)^2 \left( \frac{1}{1 - \rho^2} \right) - (0)^2 \\
= \left( \frac{1}{\sigma_2} \right)^2 \left( \frac{1}{1 - \rho^2} \right) \]

\[ I_{3,3,t} = E \left[ \frac{\partial h}{\partial \sigma_1} \frac{\partial h}{\partial \sigma_1} \right] - E \left[ \frac{\partial h}{\partial \sigma_1} \right] E \left[ \frac{\partial h}{\partial \sigma_1} \right] \\
= \frac{1}{4} \frac{1}{\sigma_1^2} \left[ \left( \frac{1}{1 - \rho^2} \right)^2 \left( 3 - 5\rho^2 + 2\rho^4 \right) \right] - \left( \frac{1}{2\sigma_1^2} \right)^2 \\
= \left( \frac{1}{1 - \rho^2} \right) \left( \frac{2 - \rho^2}{4\sigma_1^4} \right) \]

\[ I_{4,4,t} = E \left[ \frac{\partial h}{\partial \sigma_2} \frac{\partial h}{\partial \sigma_2} \right] - E \left[ \frac{\partial h}{\partial \sigma_2} \right] E \left[ \frac{\partial h}{\partial \sigma_2} \right] \\
= \frac{1}{4} \frac{1}{\sigma_2^2} \left[ \left( \frac{1}{1 - \rho^2} \right)^2 \left( 3 - 5\rho^2 + 2\rho^4 \right) \right] - \left( \frac{1}{2\sigma_2^2} \right)^2 \\
= \left( \frac{1}{1 - \rho^2} \right) \left( \frac{2 - \rho^2}{4\sigma_2^4} \right) \]

\[ I_{5,5,t} = E \left[ \frac{\partial h}{\partial \rho} \frac{\partial h}{\partial \rho} \right] - E \left[ \frac{\partial h}{\partial \rho} \right] E \left[ \frac{\partial h}{\partial \rho} \right] \\
= \left( \frac{1}{1 - \rho^2} \right)^2 \left( 2\rho^6 - 3\rho^4 + 1 \right) - \left[ - \left( \frac{\rho}{1 - \rho^2} \right) \right]^2 \\
= \left( \frac{1}{1 - \rho^2} \right)^2 \left( 1 + \rho^2 \right) \]

\[ I_{6,6,t} = E \left[ \frac{\partial h}{\partial \phi} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \phi} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \\
= (3 + 12\rho^2) - (0)^2 \\
= 3 + 12\rho^2 \]

\[ I_{1,6,t} = E \left[ \frac{\partial h}{\partial \mu_1} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \mu_1} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \\
= \left( \frac{1}{\sigma_1} \right) - (0)(0) \\
= \frac{1}{\sigma_1} \]

\[ I_{1,6,t} = \left( \frac{1}{\sigma_1} \right) - (0)(0) = \frac{1}{\sigma_1} \]
\[ I_{2,6,t} = E \left[ \frac{\partial h}{\partial \mu_2} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \mu_2} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \\
= \left( \frac{1}{\sigma_2} \right) 2\rho - (0) (0) \\
= \frac{2\rho}{\sigma_2} \\
\]

\[ I_{3,6,t} = E \left[ \frac{\partial h}{\partial \sigma_1^2} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \sigma_1^2} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \\
= (0) - \left( \frac{1}{2\sigma_1^2} \right) (0) \\
= 0 \\
\]

\[ I_{4,6,t} = E \left[ \frac{\partial h}{\partial \sigma_2^2} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \sigma_2^2} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \\
= (0) - \left( \frac{1}{2\sigma_2^2} \right) (0) \\
= 0 \\
\]

\[ I_{5,6,t} = E \left[ \frac{\partial h}{\partial \rho} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \rho} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \\
= (0) - \left( - \left( \frac{\rho}{1 - \rho^2} \right) \right) (0) \\
= 0. \\
\]

The remaining elements in the information matrix at time \( t \) are the same as that for the bivariate normal case in Section B.1.

**B.3 Coskewness and Dependence**

The expression of \( h_t \) for the generalised normal distribution used to construct the coskewness and independence joint test is

\[
h_t = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \\
+ \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2.
\]
Using the expression for the first derivatives in (56), the expectations evaluated under the null hypothesis of independent bivariate normality, are

\[
E\left[ \frac{\partial h_t}{\partial \mu_1} \right] = E \left[ \left( \frac{1}{\sigma_1} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \right] = 0
\]

\[
E\left[ \frac{\partial h_t}{\partial \mu_2} \right] = E \left[ \left( \frac{1}{\sigma_2} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] = 0
\]

\[
E\left[ \frac{\partial h_t}{\partial \sigma_1^2} \right] = E \left[ \frac{1}{2 \sigma_1^2} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \right] = \frac{1}{2 \sigma_1^2}
\]

\[
E\left[ \frac{\partial h_t}{\partial \sigma_2^2} \right] = E \left[ \frac{1}{2 \sigma_2^2} \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] = \frac{1}{2 \sigma_2^2}
\]

\[
E\left[ \frac{\partial h_t}{\partial \rho} \right] = E \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] = 0
\]

\[
E\left[ \frac{\partial h_t}{\partial \phi} \right] = E \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] = 0.
\]
as

\[
E[r_{i,t}] = \mu_i \\
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^2 \right] = 1 \\
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^1 \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^2 \right] = 0 \\
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right) \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right) \right] = 0, i \neq j.
\]

Now take expectations and evaluate the second order terms under the null hypoth-
esis of independence and bivariate normality

\[
E \left[ \left( \frac{\partial h}{\partial \mu_1} \right)^2 \right] = E \left[ \left( \frac{1}{\sigma_1} \right)^2 \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \right]
= \frac{1}{\sigma_1^2}
\]

\[
E \left[ \left( \frac{\partial h}{\partial \mu_2} \right)^2 \right] = E \left[ \left( \frac{1}{\sigma_2} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right]
= \frac{1}{\sigma_2^2}
\]

\[
E \left[ \left( \frac{\partial h}{\partial \sigma_1^2} \right)^2 \right] = E \left[ \frac{1}{4\sigma_1^4} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^4 \right]
= \frac{3}{4\sigma_1^4}
\]

\[
E \left[ \left( \frac{\partial h}{\partial \sigma_2^2} \right)^2 \right] = E \left[ \frac{1}{4\sigma_2^4} \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^4 \right]
= \frac{3}{4\sigma_2^4}
\]

\[
E \left[ \left( \frac{\partial h}{\partial \rho} \right)^2 \right] = E \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right]
= 1
\]

\[
E \left[ \left( \frac{\partial h}{\partial \phi} \right)^2 \right] = E \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^4 \right]
= 3
\]

\[
E \left[ \frac{\partial h}{\partial \mu_1} \frac{\partial h}{\partial \phi} \right] = E \left[ \left( \frac{1}{\sigma_1} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right]
= \frac{1}{\sigma_1}
\]
\[
E \left[ \frac{\partial h}{\partial \mu_2} \frac{\partial h}{\partial \phi} \right] = E \left[ \left( \frac{1}{\sigma_2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^3 \right] = 0
\]

\[
E \left[ \frac{\partial h}{\partial \sigma_1^2} \frac{\partial h}{\partial \phi} \right] = E \left[ \frac{1}{2} \frac{1}{\sigma_1^2} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^3 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] = 0
\]

\[
E \left[ \frac{\partial h}{\partial \sigma_2^2} \frac{\partial h}{\partial \phi} \right] = E \left[ \frac{1}{2} \frac{1}{\sigma_2^2} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^4 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^4 \right] = 0
\]

\[
E \left[ \frac{\partial h}{\partial \rho} \frac{\partial h}{\partial \phi} \right] = E \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^3 \right] = 0,
\]

where the following results from Appendix A are used

\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^1 \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^2 \right] = 0
\]

\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^1 \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^3 \right] = 0
\]

\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^1 \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^4 \right] = 0
\]

\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^2 \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^3 \right] = 0
\]

\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^3 \right] = 0
\]

\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^4 \right] = 3
\]

\[
E \left[ \left( \frac{r_{i,t} - \mu_i}{\sigma_i} \right)^2 \left( \frac{r_{j,t} - \mu_j}{\sigma_j} \right)^2 \right] = 1, i \neq j.
\]

Under the null hypothesis of independence and bivariate normality, the information
matrix at observation $t$ is

$$I_{1,1,t} = E \left[ \frac{\partial h}{\partial \mu_1} \frac{\partial h}{\partial \mu_1} \right] - E \left[ \frac{\partial h}{\partial \mu_1} \right] E \left[ \frac{\partial h}{\partial \mu_1} \right]$$

$$= \left( \frac{1}{\sigma_1} \right)^2 - (0)^2$$

$$= \frac{1}{\sigma_1^2}$$

$$I_{2,2,t} = E \left[ \frac{\partial h}{\partial \mu_2} \frac{\partial h}{\partial \mu_2} \right] - E \left[ \frac{\partial h}{\partial \mu_2} \right] E \left[ \frac{\partial h}{\partial \mu_2} \right]$$

$$= \left( \frac{1}{\sigma_2} \right)^2 - (0)^2$$

$$= \frac{1}{\sigma_2^2}$$

$$I_{3,3,t} = E \left[ \frac{\partial h}{\partial \sigma_1^2} \frac{\partial h}{\partial \sigma_1^2} \right] - E \left[ \frac{\partial h}{\partial \sigma_1^2} \right] E \left[ \frac{\partial h}{\partial \sigma_1^2} \right]$$

$$= \frac{3}{4} \frac{1}{\sigma_1^4} - \left( \frac{1}{2 \sigma_1^2} \right)^2$$

$$= \frac{1}{2 \sigma_1^4}$$

$$I_{4,4,t} = E \left[ \frac{\partial h}{\partial \sigma_2^2} \frac{\partial h}{\partial \sigma_2^2} \right] - E \left[ \frac{\partial h}{\partial \sigma_2^2} \right] E \left[ \frac{\partial h}{\partial \sigma_2^2} \right]$$

$$= \frac{3}{4} \frac{1}{\sigma_2^4} - \left( \frac{1}{2 \sigma_2^2} \right)^2$$

$$= \frac{1}{2 \sigma_2^4}$$

$$I_{5,5,t} = E \left[ \frac{\partial h}{\partial \rho} \frac{\partial h}{\partial \rho} \right] - E \left[ \frac{\partial h}{\partial \rho} \right] E \left[ \frac{\partial h}{\partial \rho} \right]$$

$$= 1 - (0)^2$$

$$= 1$$

$$I_{6,6,t} = E \left[ \frac{\partial h}{\partial \phi} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \phi} \right] E \left[ \frac{\partial h}{\partial \phi} \right]$$

$$= 3 - (0)^2$$

$$= 3$$
\[ I_{1,6,t} = E \left[ \frac{\partial h}{\partial \mu_1} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \mu_1} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \]
\[ = \left( \frac{1}{\sigma_1} \right) - (0) (0) \]
\[ = \frac{1}{\sigma_1} \]

\[ I_{2,6,t} = E \left[ \frac{\partial h}{\partial \mu_2} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \mu_2} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \]
\[ = 0 - (0) (0) \]
\[ = 0 \]

\[ I_{3,6,t} = E \left[ \frac{\partial h}{\partial \sigma_1^2} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \sigma_1^2} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \]
\[ = (0) - \left( \frac{1}{2\sigma_1^2} \right) (0) \]
\[ = 0 \]

\[ I_{4,6,t} = E \left[ \frac{\partial h}{\partial \sigma_2^2} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \sigma_2^2} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \]
\[ = (0) - \left( \frac{1}{2\sigma_2^2} \right) (0) \]
\[ = 0 \]

\[ I_{5,6,t} = E \left[ \frac{\partial h}{\partial \rho} \frac{\partial h}{\partial \phi} \right] - E \left[ \frac{\partial h}{\partial \rho} \right] E \left[ \frac{\partial h}{\partial \phi} \right] \]
\[ = (0) - (0) (0) \]
\[ = 0. \]

The other elements in the information matrix at observation t are the same as in the bivariate normal example in Appendix B.1.
C Derivation of Test Statistics

C.1 Coskewness Test

Consider the following generalisation of the bivariate normal distribution

\[
f(r_{1,t}, r_{2,t}) = \exp \left[ -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right], \tag{57}\]

where

\[
\eta = \ln \int \int \exp \left[ -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] dr_1 dr_2
\]

\[
= \ln \int \int \exp [h] dr_1 dr_2, \tag{58}\]

and

\[
h = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2. \tag{59}\]

A test of bivariate normality is based on the null hypothesis

\[H_0 : \phi = 0.\tag{60}\]

Under the null hypothesis of bivariate normality the maximum likelihood estimators of the unknown parameters are simply

\[
\hat{\mu}_i = \frac{1}{T} \sum_t r_{i,t}; \hat{\sigma}_i^2 = \frac{1}{T} \sum_t (r_{i,t} - \hat{\mu}_i)^2; \hat{\rho} = \frac{1}{T} \sum_t \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right) \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right). \tag{61}\]
Let the parameters of (57) be

\[ \theta = \{ \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho, \phi \} . \]

The log-likelihood at time \( t \) is

\[
\ln L_t = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] \\
-2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \\
+\phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - \eta \\
= h_t - \eta.
\]

where

\[
h_t = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] \\
+\phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2,
\]

and \( \eta \) is given by (58).

Using equation (17) and the results from Appendix B, the information matrix under \( H_0 \) is

\[
I = -E \left[ \frac{\partial^2 \ln L}{\partial \Theta_i \partial \Theta_j} \right]_{\phi=0}
\]

\[
= \begin{pmatrix}
\frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1 \sigma_2} & 0 & 0 & 0 & \frac{(1 - \rho^2)}{\sigma_1} \\
-\frac{\rho}{\sigma_1 \sigma_2} & \frac{1}{\sigma_2^2} & 0 & 0 & 0 & \frac{2\rho (1 - \rho^2)}{\sigma_2} \\
0 & 0 & \frac{2 - \rho^2}{4 \sigma_1^4} & \frac{\rho^2}{4 \sigma_1^2 \sigma_2^2} & \frac{\rho}{2 \sigma_1^2} & 0 \\
0 & 0 & \frac{-\rho^2}{4 \sigma_1^2 \sigma_2^2} & \frac{2 - \rho^2}{2 \sigma_1^2} & \frac{\rho}{2 \sigma_1} & 0 \\
0 & 0 & \frac{-\rho}{2 \sigma_1^2} & \frac{-\rho}{2 \sigma_2^2} & \frac{1 + \rho^2}{1 - \rho^2} & 0 \\
(1 - \rho^2) & \frac{2\rho (1 - \rho^2)}{\sigma_1} & 0 & 0 & 0 & (3 + 12\rho^2) (1 - \rho^2) \end{pmatrix}.
\]
so that

\[
I^{-1} = \left( \frac{1 - \rho^2}{T} \right) \begin{bmatrix}
\frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1 \sigma_2} & 0 & 0 & 0 & \frac{(1 - \rho^2)}{\sigma_1} \\
-\frac{\rho}{\sigma_1 \sigma_2} & \frac{1}{\sigma_2^2} & 0 & 0 & 0 & \frac{2 \rho}{\sigma_2} (1 - \rho^2) \\
0 & 0 & \frac{2 - \rho^2}{4 \sigma_1^2} & -\frac{\rho^2}{2 \sigma_1^2} & -\frac{\rho}{2 \sigma_1} & 0 \\
0 & 0 & -\frac{\rho^2}{4 \sigma_1^2 \sigma_2^2} & \frac{2 - \rho^2}{4 \sigma_2^2} & \frac{\rho}{2 \sigma_2} & 0 \\
0 & 0 & \frac{(1 - \rho^2)}{\sigma_1} & -\frac{\rho}{2 \sigma_1^2} & -\frac{\rho}{2 \sigma_2^2} & 1 + \rho^2 \\
\frac{2 \rho (1 - \rho^2)}{\sigma_2} & 0 & 0 & 0 & 0 & (3 + 12 \rho^2) (1 - \rho^2)
\end{bmatrix}^{-1}.
\]

Evaluating the gradient for \( \phi \) under the null gives

\[
\frac{\partial \ln L}{\partial \phi} = \sum_{t=1}^{T} \left( \frac{\partial h}{\partial \phi} \right) - T \left( \frac{\partial \eta}{\partial \phi} \right) = \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) - T \left[ E \left( \frac{\partial h}{\partial \phi} \right) \right]
\]

\[
= \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - T (0)
\]

\[= \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2. \tag{64}\]

The gradient vector under \( H_0 \) is

\[
G = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2
\end{bmatrix}'. \tag{65}\]

The Lagrange multiplier statistic is obtained by substituting (63) and (65) into

\[
LM = G' I^{-1} G, \tag{66}\]

and replacing the unknown population parameters by consistent estimators under the
null hypothesis. The pertinent statistic is

$$LM = G' I^{-1} G,$$

$$= - \left( \frac{1-\tilde{\rho}^2}{-2\tilde{\rho}^2 + 4\tilde{\rho}^4 - 2} \right) \frac{1}{T} \left( \sum_{t=1}^{T} \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right) \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right) \right)^2$$

$$= \frac{T}{4\tilde{\rho}^2 + 2} \left( \frac{1}{T} \sum_{t=1}^{T} \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right) \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right) \right)^2,$$  \hspace{1cm} (67)

which is asymptotically distributed as $\chi^2_1$ under the null hypothesis.

### C.2 Coskewness and Dependence Joint Test

A joint test of coskewness and dependence is given by the restrictions

$$H_0 : \phi = 0; \rho = 0,$$

in (57). Using the results from Appendix B, the information matrix under $H_0$ is

$$I = -E \left[ \frac{\partial^2 \ln L}{\partial \Theta_i \partial \Theta_j} \right]_{\phi=\rho=0}$$

$$= T \left[ \begin{array}{cccccc}
\frac{1}{\sigma_1^2} & 0 & 0 & 0 & 0 & \frac{1}{\sigma_1} \\
0 & \frac{1}{\sigma_2^2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2\sigma_1^4} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2\sigma_2^4} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\frac{1}{\sigma_1^2} & 0 & 0 & 0 & 0 & 3 \\
\end{array} \right],$$

so that
\[
I^{-1} = \frac{1}{T} \begin{bmatrix}
\frac{1}{\sigma_1^2} & 0 & 0 & 0 & \frac{1}{\sigma_1} \\
0 & \frac{1}{\sigma_2^2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2\sigma_1^2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2\sigma_2^4} & 0 \\
\frac{1}{\sigma_1} & 0 & 0 & 0 & 3
\end{bmatrix}^{-1}.
\]

Evaluating the gradients for \( \phi \) and \( \rho \) under the null yields

\[
\frac{\partial \ln L}{\partial \phi} = \sum_{t=1}^{T} \left( \frac{\partial h_t}{\partial \phi} \right) - T \left( \frac{\partial \eta}{\partial \phi} \right) \\
= \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - T \left[ E \left( \frac{\partial h}{\partial \phi} \right) \right] \\
= \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - T(0)
\]
and

\[
\frac{\partial \ln L}{\partial \rho} = \sum_{t=1}^{T} \left( \frac{\partial h_t}{\partial \rho} \right) - T \left( \frac{\partial \eta}{\partial \rho} \right) \\
= \sum_{t=1}^{T} \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] - T \left[ E \left( \frac{\partial h}{\partial \rho} \right) \right] \\
= \sum_{t=1}^{T} \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] - T(0) \\
= \sum_{t=1}^{T} \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right].
\]

The gradient vector under \( H_0 \) is thus

\[
G = \begin{bmatrix}
0 & 0 & 0 & \sum_{t=1}^{T} \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) \right] & \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2
\end{bmatrix}.
\]
The Lagrange multiplier statistic is obtained by substituting (68) and (69) into (66) and replacing the unknown population parameters by consistent estimators under the null hypothesis as given in (61)

\[
LM = G^T G
\]

\[
= T \left( \frac{1}{T} \sum_{t=1}^{T} \left[ \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right) \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right) \right]^2 + \right.
\]

\[
\left. \frac{1}{2} \left( \frac{1}{T} \sum_{t=1}^{T} \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right)^2 + \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right)^2 \right) \right)^2
\]

which is asymptotically distributed as \( \chi^2_2 \) under the null hypothesis.

**C.3 Derivation of \( FR_2 \)**

In this Appendix the pertinent variance used in the \( FR_2 \) test is derived. Consider

\[
Var \left( \hat{\nu}_{y|x,i} - \hat{\rho}_x \right) = Var \left( \hat{\nu}_{y|x,i} \right) + Var \left( \hat{\rho}_x \right) - 2 Cov \left( \hat{\nu}_{y|x,i}, \hat{\rho}_x \right),
\]

where

\[
\hat{\nu}_{y|x,i} = \frac{\hat{\rho}_y}{\sqrt{1 + \hat{\delta} \left( 1 - \hat{\rho}_y \right)}}
\]

\[
\hat{\delta} = \left( \frac{s_{y,i}^2 - s_{x,i}^2}{s_{x,i}^2} \right).
\]

The first term is

\[
Var \left( \hat{\nu}_{y|x,i} \right) = Var \left\{ \frac{\hat{\rho}_y}{\sqrt{1 + \left( \frac{s_{y,i}^2 - s_{x,i}^2}{s_{x,i}^2} \right) \left( 1 - \hat{\rho}_y \right)}} \right\}
\]

\[
= Var \left\{ \frac{m_{11,y}}{\sqrt{m_{20,y} m_{20,y}}} \right\}
\]

\[
= Var \left\{ \frac{m_{20,y} - m_{20,x}}{m_{20,x}} \left( 1 - \frac{m_{11,y}}{m_{20,y} m_{20,y}} \right) \right\}
\]

\[
= Var \left\{ \frac{m_{20,y} m_{20,x} - m_{20,y} + 1}{m_{11,y} m_{20,x}} \right\}^{-\frac{1}{2}}
\]

67
where

\[ m_{ij,k} = \sum_{t} \left[ \frac{(r_{1,t} - \mu_{r_1})^i (r_{2,t} - \mu_{r_2})^j}{k} \right]. \]

\[
\begin{align*}
  & = \frac{1}{4} \left[ \frac{-\mu_{r_2}}{2} \sigma_{11,y} \mu_{20,x} + \frac{\mu_{r_2}}{2} \sigma_{11,y} \mu_{20,x} \right] \left[ \frac{2\mu_{r_2} \sigma_{02,y}^2}{\mu_{11,y}^2 \mu_{20,x}} - 1 \right] Var(m_{20,y}) + \\
  & \quad \left[ \frac{\mu_{r_2}}{2} \sigma_{11,y} \mu_{20,x} \right]^2 Var(m_{02,y}) + \left[ \frac{-2\mu_{r_2} \sigma_{02,y}^2}{\mu_{11,y}^2 \mu_{20,x}} \right]^2 Var(m_{11,y}) + \\
  & \quad \left[ \frac{-\mu_{r_2}}{2} \sigma_{11,y} \mu_{20,x} + \frac{\mu_{r_2}}{2} \sigma_{11,y} \mu_{20,x} \right]^2 Var(m_{20,x}) + \\
  & \quad 2 \left[ \frac{2\mu_{r_2} \sigma_{02,y}^2}{\mu_{11,y}^2 \mu_{20,x}} - 1 \right] \left[ \frac{\mu_{r_2}}{2} \sigma_{11,y} \mu_{20,x} \right] Cov(m_{20,y}, m_{02,y}) + \\
  & \quad 2 \left[ \frac{2\mu_{r_2} \sigma_{02,y}^2}{\mu_{11,y}^2 \mu_{20,x}} - 1 \right] \left[ \frac{-2\mu_{r_2} \sigma_{02,y}^2}{\mu_{11,y}^2 \mu_{20,x}} \right] Cov(m_{20,y}, m_{11,y}) + \\
  & \quad 2 \left[ \frac{\mu_{r_2}}{2} \sigma_{11,y} \mu_{20,x} \right] \left[ \frac{-2\mu_{r_2} \sigma_{02,y}^2}{\mu_{11,y}^2 \mu_{20,x}} \right] Cov(m_{02,y}, m_{11,y}) \right] \}
\]

where

\[
Var(m_{20,y}) = \frac{1}{T_y} \left[ \mu_{40,y} - \sigma_{20,y}^2 \right],
\]

\[
Var(m_{02,y}) = \frac{1}{T_y} \left[ \mu_{4e,y} - \sigma_{02,y}^2 \right],
\]

\[
Var(m_{11,y}) = \frac{1}{T_y} \left[ \mu_{22,y} - \sigma_{11,y}^2 \right],
\]

\[
Var(m_{20,x}) = \frac{1}{T_x} \left[ \mu_{40,x} - \sigma_{20,x}^2 \right],
\]

\[
Cov(m_{20,y}, m_{02,y}) = \frac{1}{T_y} \left[ \mu_{22,y} - \mu_{20,y} \mu_{02,y} \right],
\]

\[
Cov(m_{20,y}, m_{11,y}) = \frac{1}{T_y} \left[ \mu_{31,y} - \mu_{20,y} \mu_{11,y} \right],
\]

\[
Cov(m_{02,y}, m_{11,y}) = \frac{1}{T_y} \left[ \mu_{13,y} - \mu_{02,y} \mu_{11,y} \right].
\]
and

\[ \begin{align*} 
\mu_{20,y} &= \sigma_{1,y}^2, \\
\mu_{02,y} &= \sigma_{2,y}^2, \\
\mu_{31,y} &= 3\rho_y^2 \sigma_{1,y}^2 \sigma_{2,y}, \\
\mu_{13,y} &= 3\rho_y \sigma_{1,y}^2 \sigma_{2,y}^2, \\
\mu_{40,y} &= 3\sigma_{1,y}^4, \\
\mu_{04,y} &= 3\sigma_{2,y}^4, \\
\mu_{11,y} &= \rho_y \sigma_{1,y} \sigma_{2,y}, \\
\mu_{22,y} &= (1 + 2\rho_y^2) \sigma_{1,y}^2 \sigma_{2,y}^2, \\
\mu_{20,x} &= \sigma_{1,x}^2; \quad \mu_{02,x} = \sigma_{2,x}^2; \quad \mu_{40,x} = 3\sigma_{1,x}^4; \\
\delta &\equiv \frac{\mu_{20,y}}{\mu_{20,x}} - 1 = \frac{\sigma_{1,y}^2}{\sigma_{1,x}^2} - 1. 
\end{align*} \]

together with

\[ \begin{align*} 
\mu_{20,x} &= \sigma_{1,x}^2; \quad \mu_{02,x} = \sigma_{2,x}^2; \quad \mu_{40,x} = 3\sigma_{1,x}^4; \\
\delta &\equiv \frac{\mu_{20,y}}{\mu_{20,x}} - 1 = \frac{\sigma_{1,y}^2}{\sigma_{1,x}^2} - 1. 
\end{align*} \]

Substituting these expressions into \( \text{Var} \left( \hat{\nu}_{y|x} \right) \) yields

\[ \text{Var} \left( \hat{\nu}_{y|x} \right) = \left[ \frac{1}{2} \left( \frac{(1 + \delta)^2}{1 + \delta (1 - \rho_y^2)} \right)^3 \left( \frac{1}{T_y} \left( (2 - \rho_y^2) (1 - \rho_y^2)^2 \right) + \frac{1}{T_x} \left( \rho_y^2 (1 - \rho_y^2)^2 \right) \right) \right]. \]

The second term \( \text{Var} \left( \hat{\rho}_x \right) \), is derived earlier in the equation for \( I_{5,5,t} \) in Appendix B.1 and is given by

\[ \text{Var} \left( \hat{\rho}_x \right) = \frac{1}{T_x} (1 - \rho_x^2)^2. \]
The third and last term is

$$\text{Cov} (\hat{\nu}_{y|x}, \hat{\rho}_x) = \text{Cov} \left\{ \frac{\hat{\rho}_y}{\sqrt{1 + \left( \frac{s_{y,i}^2 - s_{x,i}^2}{s_{y,i}^2} \right) (1 - \hat{\rho}_y)}} \right\}$$

$$= \text{Cov} \left\{ \frac{m_{11,y}}{\sqrt{m_{20,y}m_{02,y}}} \right\} \frac{m_{11,x}}{\sqrt{m_{20,x}m_{02,x}}}$$

$$= \text{Cov} \left\{ \left( \frac{m_{20,y}m_{02,y}}{m_{11,y}m_{20,x}} - 1 \right)^{-\frac{1}{2}}, \left( \frac{m_{20,x}m_{02,x}}{m_{11,x}} \right)^{-\frac{1}{2}} \right\}$$

$$= \left[ \begin{array}{cc}
-\frac{1}{4} & 1 \\
1 & -\frac{1}{4}
\end{array} \right] \left[ \begin{array}{cc}
-\frac{\mu_{20,y}^2}{\mu_{11,y}^2}, \mu_{02,y}^2 \\
\mu_{11,y}^2, \mu_{20,y}^2
\end{array} \right] \text{Var} (m_{20,x}) +$$

$$\left[ \begin{array}{cc}
-\frac{\mu_{20,y}^2}{\mu_{11,y}^2}, \mu_{02,y}^2 \\
\mu_{11,y}^2, \mu_{20,y}^2
\end{array} \right] \left[ \begin{array}{cc}
-\frac{\mu_{11,x}^2}{\mu_{02,x}^2} \\
\mu_{02,x}^2
\end{array} \right] \text{Cov} (m_{20,x}, m_{02,x}) +$$

$$\left[ \begin{array}{cc}
-\frac{\mu_{20,y}^2}{\mu_{11,y}^2}, \mu_{02,y}^2 \\
\mu_{11,y}^2, \mu_{20,y}^2
\end{array} \right] \left[ \begin{array}{cc}
2\mu_{11,x}^2 \\
\mu_{02,x}^2
\end{array} \right] \text{Cov} (m_{20,x}, m_{11,x}) ,$$

where

$$\text{Var} (m_{20,x}) = \frac{1}{T_x} \left[ \mu_{40,x} - \mu_{20,x}^2 \right] ;$$

$$\text{Cov} (m_{20,x}, m_{02,x}) = \frac{1}{T_x} \left[ \mu_{22,x} - \mu_{20,x} \mu_{02,x} \right] ;$$

$$\text{Cov} (m_{20,x}, m_{11,x}) = \frac{1}{T_x} \left[ \mu_{31,x} - \mu_{20,x} \mu_{11,x} \right] ;$$

and

$$\mu_{31,y} = 3\rho_x \sigma_{1,x}^2 \sigma_{2,x}; \ \mu_{11,x} = \rho_x \sigma_{1,x} \sigma_{2,x}; \ \mu_{22,x} = (1 + 2\rho_x^2) \sigma_{1,x}^2 \sigma_{2,x}^2;$$
use the results of Appendix A. In which case

\[
\text{Cov}(\hat{\rho}_{y|x}, \hat{\rho}_x) = \frac{1}{2T_x} \frac{\rho_y \rho_x}{\sqrt{[1 + \delta (1 - \rho_y^2)]^3}} \left( 1 - \rho_y^2 \right) \left( 1 - \rho_x^2 \right) (1 + \delta).
\]
D Supplementary Empirical Results

This appendix contains a number of additional empirical results.

Table 8 gives the results of testing for independence based on testing that the correlation coefficient amongst pairs of asset returns is zero, during the noncrisis, crisis and total sample periods. The results show strong evidence of dependence both within countries, across countries and across asset market classes. The one exception is between the Japanese real estate market and the equity market for Singapore in the noncrisis period.

Tables 9 presents the results of the joint test of independence and no coskewness amongst the pairs of asset markets, using (36). The results show strong rejection of the null in all cases with the exception of the Japanese equity market and the real estate market in Singapore.

Tables 10 to 13 contain the first and third versions of the contemporaneous and dynamic coskewness tests of contagion using $CS_1$ and $CS_3$. The contemporaneous statistics are based on equations (47) and (51) where the source country is the level term in the coskewness statistic, and (48) and (52) where the source country is the squared term in the coskewness statistic. The dynamic coskewness contagion tests are based on

\[
CS_1 (i \to j; r_{i,t-1}^1, r_{j,t}^2) = \left( \frac{\hat{\psi}_y (r_{i,t-1}^1, r_{j,t}^2) - \hat{\psi}_z (r_{i,t-1}^1, r_{j,t}^2)}{4 \hat{\psi}_y |_z (r_{i,t-1}^1, r_{j,t}^2) + 2 \frac{4 \hat{\psi}_z^2 (r_{i,t-1}^1, r_{j,t}) + 2}{T_y}} \right)^2
\]

\[
CS_1 (i \to j; r_{i,t-1}^2, r_{j,t}^1) = \left( \frac{\hat{\psi}_y (r_{i,t-1}^2, r_{j,t}^1) - \hat{\psi}_z (r_{i,t-1}^2, r_{j,t}^1)}{4 \hat{\psi}_y |_z (r_{i,t-1}^2, r_{j,t}) + 2 \frac{4 \hat{\psi}_z^2 (r_{i,t-1}^2, r_{j,t}) + 2}{T_z}} \right)^2
\]
and

\[ CS_3 (i \rightarrow j; r_{i,t-1,2}^1, r_{j,t}^2) = \left( \frac{\hat{\psi}_y (r_{i,t-1,2}^1, r_{j,t}^2) - \hat{\psi}_z (r_{i,t-1,2}^1, r_{j,t}^2)}{\sqrt{\text{Var} (\hat{\psi}_y (r_{i,t-1,1}^1, r_{j,t}^2) - \hat{\psi}_z (r_{i,t-1,1}^1, r_{j,t}^2))}} \right)^2 \]  

(73)

\[ CS_3 (i \rightarrow j; r_{i,t-1,1}^2, r_{j,t}^1) = \left( \frac{\hat{\psi}_y (r_{i,t-1,2}^1, r_{j,t}^2) - \hat{\psi}_z (r_{i,t-1,2}^1, r_{j,t}^2)}{\sqrt{\text{Var} (\hat{\psi}_y (r_{i,t-1,1}^1, r_{j,t}^2) - \hat{\psi}_z (r_{i,t-1,1}^1, r_{j,t}^2))}} \right)^2 \]  

(74)
Table 8:
Tests of independence on adjusted returns, based on the correlation coefficient. The test statistic is asymptotically distributed as chi-squared with one degree of freedom. The 5% critical value is 3.84.

<table>
<thead>
<tr>
<th>((x_j^1))</th>
<th>(\text{Real Estate})</th>
<th>(\text{Equity})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HK</td>
<td>JP</td>
</tr>
<tr>
<td>HK</td>
<td>n.a.</td>
<td>17.86</td>
</tr>
<tr>
<td>Real Estate</td>
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<td>4.13</td>
</tr>
<tr>
<td>JP</td>
<td>n.a.</td>
<td>51.39</td>
</tr>
<tr>
<td>SG</td>
<td>n.a.</td>
<td>52.26</td>
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</table>

<table>
<thead>
<tr>
<th>((y_j^1))</th>
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<th>(\text{Equity})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HK</td>
<td>JP</td>
</tr>
<tr>
<td>HK</td>
<td>n.a.</td>
<td>13.21</td>
</tr>
<tr>
<td>Real Estate</td>
<td>n.a.</td>
<td>8.88</td>
</tr>
<tr>
<td>JP</td>
<td>n.a.</td>
<td>60.64</td>
</tr>
<tr>
<td>SG</td>
<td>n.a.</td>
<td>22.26</td>
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<table>
<thead>
<tr>
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<tbody>
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<td></td>
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<td>JP</td>
</tr>
<tr>
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<td>n.a.</td>
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<tr>
<td>Real Estate</td>
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<tr>
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<td>SG</td>
<td>n.a.</td>
<td>72.48</td>
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<table>
<thead>
<tr>
<th>((z_j^1))</th>
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<th>(\text{Equity})</th>
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</thead>
<tbody>
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<td>HK</td>
<td>JP</td>
</tr>
<tr>
<td>HK</td>
<td>n.a.</td>
<td>72.48</td>
</tr>
<tr>
<td>Equity</td>
<td>n.a.</td>
<td>45.84</td>
</tr>
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74
Table 9:
Joint test of independence and no coskewness on adjusted returns, based on (36).
The test statistic is asymptotically distributed as chi-squared with two degrees of freedom. The 5% critical value is 5.99.

<table>
<thead>
<tr>
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<th>((x_2^2))</th>
<th>((z_1^2))</th>
<th>((z_2^2))</th>
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<td>Real Estate</td>
<td>Equity</td>
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<tr>
<td></td>
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<td>JP</td>
<td>SG</td>
<td>HK</td>
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<tr>
<td>Noncrisis</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>HK</td>
<td>n.a.</td>
<td>23.78</td>
<td>85.91</td>
<td>410.19</td>
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<td>18.19</td>
<td>n.a.</td>
<td>4.33</td>
<td>12.94</td>
</tr>
<tr>
<td>SG</td>
<td>67.52</td>
<td>6.89</td>
<td>n.a.</td>
<td>53.96</td>
</tr>
<tr>
<td>Equity</td>
<td></td>
<td></td>
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<td></td>
</tr>
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<td>HK</td>
<td>401.65</td>
<td>15.09</td>
<td>66.84</td>
<td>n.a.</td>
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<td>16.74</td>
<td>188.84</td>
<td>3.86</td>
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</tr>
<tr>
<td>SG</td>
<td>45.99</td>
<td>5.60</td>
<td>155.23</td>
<td>82.57</td>
</tr>
<tr>
<td>Crisis</td>
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<td>HK</td>
<td>n.a.</td>
<td>19.43</td>
<td>130.26</td>
<td>158.47</td>
</tr>
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<td>n.a.</td>
<td>13.55</td>
<td>16.43</td>
</tr>
<tr>
<td>SG</td>
<td>146.38</td>
<td>13.34</td>
<td>n.a.</td>
<td>91.57</td>
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<tr>
<td>Equity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HK</td>
<td>149.67</td>
<td>8.62</td>
<td>62.11</td>
<td>n.a.</td>
</tr>
<tr>
<td>Real Estate JF</td>
<td>15.11</td>
<td>112.07</td>
<td>7.18</td>
<td>22.61</td>
</tr>
<tr>
<td>SG</td>
<td>90.17</td>
<td>10.35</td>
<td>136.78</td>
<td>76.82</td>
</tr>
<tr>
<td>Total</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>HK</td>
<td>n.a.</td>
<td>46.94</td>
<td>372.64</td>
<td>505.77</td>
</tr>
<tr>
<td>Real Estate JF</td>
<td>52.84</td>
<td>n.a.</td>
<td>28.15</td>
<td>40.12</td>
</tr>
<tr>
<td>SG</td>
<td>465.39</td>
<td>32.60</td>
<td>n.a.</td>
<td>270.41</td>
</tr>
<tr>
<td>Equity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HK</td>
<td>505.31</td>
<td>28.02</td>
<td>170.08</td>
<td>n.a.</td>
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<tr>
<td>Real Estate JF</td>
<td>38.88</td>
<td>327.48</td>
<td>14.99</td>
<td>72.85</td>
</tr>
<tr>
<td>SG</td>
<td>219.17</td>
<td>720.68</td>
<td>397.63</td>
<td>230.44</td>
</tr>
</tbody>
</table>
Table 10:
Contemporaneous coskewness tests of contagion on adjusted returns, using (47) and (51). The source country is the level term in the coskewness statistic. The 5% critical value is 3.84.

\[
CS_1 (i \rightarrow j; r^1_i, r^2_j) \text{ test} \\
\begin{array}{c|ccc|ccc}
\multirow{2}{*}{\text{Source (}r^1_i\text{)}} & \multicolumn{3}{c|}{\text{Real Estate}} & \multicolumn{3}{c}{\text{Equity}} \\
& HK & JP & SG & HK & JP & SG \\
\hline
\text{Recipient (}r^2_j\text{)} & HK & n.a. & 0.63 & 0.19 & 1.79 & 0.13 & 1.28 \\
 & Real Estate & JP & 0.70 & n.a. & 0.35 & 0.47 & 0.68 & 0.45 \\
 & & SG & 0.01 & 0.05 & n.a. & 0.04 & 0.06 & 0.03 \\
 & Equity & JP & 2.09 & 0.45 & 1.08 & n.a. & 0.08 & 1.79 \\
 & & SG & 0.04 & 0.28 & 0.04 & 0.05 & n.a. & 0.04 \\
 & & & & 1.14 & 0.42 & 0.24 & 0.95 & 0.00 & n.a. \\
\end{array}
\]

\[
CS_3 (i \rightarrow j; r^1_i, r^2_j) \text{ test} \\
\begin{array}{c|ccc|ccc}
\multirow{2}{*}{\text{Source (}r^1_i\text{)}} & \multicolumn{3}{c|}{\text{Real Estate}} & \multicolumn{3}{c}{\text{Equity}} \\
& HK & JP & SG & HK & JP & SG \\
\hline
\text{Recipient (}r^2_j\text{)} & HK & n.a. & 1.11 & 0.33 & 3.12 & 0.23 & 2.25 \\
 & Real Estate & JP & 1.23 & n.a. & 0.62 & 0.83 & 1.20 & 0.80 \\
 & & SG & 0.01 & 0.09 & n.a. & 0.07 & 0.10 & 0.05 \\
 & Equity & JP & 3.63 & 0.79 & 1.87 & n.a. & 0.13 & 3.11 \\
 & & SG & 0.07 & 0.49 & 0.08 & 0.09 & n.a. & 0.06 \\
 & & & & 1.99 & 0.75 & 0.42 & 1.65 & 0.00 & n.a. \\
\end{array}
\]
Table 11:
Contemporaneous coskewness tests of contagion on adjusted returns, using (48) and (52). The source country is the squared term in the coskewness statistic. The 5% critical value is 3.84.

<table>
<thead>
<tr>
<th>Source $(r_i^2)$</th>
<th>Real Estate</th>
<th>Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>HK</td>
<td>n.a. 0.69 0.01</td>
<td></td>
</tr>
<tr>
<td>JP</td>
<td>0.63 n.a. 0.05</td>
<td></td>
</tr>
<tr>
<td>SG</td>
<td>0.19 0.35 n.a.</td>
<td></td>
</tr>
<tr>
<td>HK</td>
<td>1.81 0.47 0.04</td>
<td></td>
</tr>
<tr>
<td>Equity</td>
<td>0.13 0.70 0.06</td>
<td></td>
</tr>
<tr>
<td>SG</td>
<td>1.30 0.45 0.03</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source $(r_i^2)$</th>
<th>Real Estate</th>
<th>Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>HK</td>
<td>n.a. 1.22 0.01</td>
<td></td>
</tr>
<tr>
<td>JP</td>
<td>1.12 n.a. 0.09</td>
<td></td>
</tr>
<tr>
<td>SG</td>
<td>0.33 0.62 n.a.</td>
<td></td>
</tr>
<tr>
<td>HK</td>
<td>3.14 0.82 0.07</td>
<td></td>
</tr>
<tr>
<td>Equity</td>
<td>0.23 1.23 0.11</td>
<td></td>
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<tr>
<td>SG</td>
<td>2.27 0.80 0.05</td>
<td></td>
</tr>
</tbody>
</table>

$CS_1(i \rightarrow j; r_i^2, r_j^1)$ test

$CS_3(i \rightarrow j; r_i^2, r_j^1)$ test
Table 12:
Dynamic coskewness tests of contagion on adjusted returns, using (71) and (73). The source country is the level term in the coskewness statistic. The 5% critical value is 3.84.

\[ CS_1 (i \rightarrow j; r_{i,t-1}^1, r_{j,t}^2) \] test

<table>
<thead>
<tr>
<th>Recipient ( (r_{j,t}^2) )</th>
<th></th>
<th></th>
<th>Source ( (r_{i,t-1}^1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Real Estate</td>
<td></td>
<td>Equity</td>
</tr>
<tr>
<td></td>
<td>HK</td>
<td>JP</td>
<td>SG</td>
</tr>
<tr>
<td>Real Estate</td>
<td>n.a.</td>
<td>0.22</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td>0.17</td>
<td>n.a.</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>2.42</td>
<td>0.21</td>
<td>n.a.</td>
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<tr>
<td>Equity</td>
<td>0.92</td>
<td>0.25</td>
<td>1.00</td>
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<tr>
<td></td>
<td>0.17</td>
<td>1.86</td>
<td>0.16</td>
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<tr>
<td></td>
<td>1.40</td>
<td>0.13</td>
<td>0.49</td>
</tr>
</tbody>
</table>

\[ CS_3 (i \rightarrow j; r_{i,t-1}^1, r_{j,t}^2) \] test

<table>
<thead>
<tr>
<th>Recipient ( (r_{j,t}^2) )</th>
<th></th>
<th></th>
<th>Source ( (r_{i,t-1}^1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Real Estate</td>
<td></td>
<td>Equity</td>
</tr>
<tr>
<td></td>
<td>HK</td>
<td>JP</td>
<td>SG</td>
</tr>
<tr>
<td>Real Estate</td>
<td>n.a.</td>
<td>0.39</td>
<td>2.21</td>
</tr>
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<td></td>
<td>0.30</td>
<td>n.a.</td>
<td>1.10</td>
</tr>
<tr>
<td></td>
<td>4.28</td>
<td>0.37</td>
<td>n.a.</td>
</tr>
<tr>
<td>Equity</td>
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<td>0.44</td>
<td>1.78</td>
</tr>
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<td>0.30</td>
<td>3.29</td>
<td>0.28</td>
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<td>2.47</td>
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</table>
Table 13:
Dynamic coskewness tests of contagion on adjusted returns, using (72) and (74). The source country is the squared term in the coskewness statistic. The 5% critical value is 3.84.

<table>
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<tr>
<th>Source $(r_{i,t-1}^2)$</th>
<th>Real Estate</th>
<th>Equity</th>
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</thead>
<tbody>
<tr>
<td>HK</td>
<td>n.a. 0.35</td>
<td>1.16</td>
</tr>
<tr>
<td>JP</td>
<td>0.31 n.a.</td>
<td>0.14</td>
</tr>
<tr>
<td>SG</td>
<td>0.00 0.98</td>
<td>n.a.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CS$<em>1$ $(i \rightarrow j; r</em>{i,t-1}^2, r_{j,t}^1)$ test</th>
<th>Source $(r_{i,t-1}^2)$</th>
</tr>
</thead>
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<tr>
<td>Recipient $(r_{j,t}^1)$</td>
<td>Real Estate</td>
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<tr>
<td>Real Estate</td>
<td>0.54 n.a.</td>
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<tr>
<td>JP</td>
<td>0.00 0.98</td>
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<tr>
<td>SG</td>
<td>0.00 1.62</td>
</tr>
</tbody>
</table>

<table>
<thead>
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<th>CS$<em>3$ $(i \rightarrow j; r</em>{i,t-1}^2, r_{j,t}^1)$ test</th>
<th>Source $(r_{i,t-1}^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recipient $(r_{j,t}^1)$</td>
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</tr>
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<tr>
<td>Real Estate</td>
<td>0.00 2.87</td>
</tr>
<tr>
<td>JP</td>
<td>0.00 1.74</td>
</tr>
<tr>
<td>SG</td>
<td>0.00 1.74</td>
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References


*IMF Staff Papers*, 46, 167-195.


*Journal of Business*, 78, 39-69.

*Communications in Statistics-Theor and Methods*, 12, 103-117.


