Expectations of linear functions with respect to truncated multinormal distributions — With applications for uncertainty analysis in environmental modelling

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Abstract

This paper discusses results concerning multivariate normal distributions that are subject to truncation by a hyperplane and how such results can be applied to uncertainty analysis in the environmental sciences. We present a suite of results concerning truncated multivariate normal distributions, some of which already appear in the mathematical literature. The focus here is to make these types of results more accessible to the environmental science community and to this end we include a conceptually simple alternative derivation of an important result. We illustrate how the theory of truncated multivariate normal distributions can be employed in the environmental sciences by means of an example from the economics of climate change control.

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1. Introduction

A problem that arises in a number of applications in environmental modelling is the need to find the expectation of a function of many uncertain variables, subject to some kind of sharp cut-off, threshold or truncation limit. For example, beyond a certain climatic limit, snowfall may suddenly become zero; beyond geographic limits like coastlines, rivers or mountain ranges, the population densities of many species suddenly become zero; and beyond some economic limits, firms or countries may suddenly choose to opt out of some voluntary environmental scheme, because it is unprofitable to them.

The functions in these applications will usually in fact be non-linear, and also the underlying probability distributions will usually be non-normal. However, within an acceptable margin of error, owing to the Central Limit Theorem (Rohatgi and Ehsanes Saleh, 2001), one can often linearise the functions and assume a multivariate normal (multinormal) distribution of random variables. For example, the latter assumption is often employed when the random variables represent errors in a model or uncertainties in a process.

In environmental modelling, uncertainty is typically dealt with by solving a deterministic model under a number of alternative realisations of the random variables, using Monte Carlo experiments, Gaussian quadrature or other procedures where expectations of functions of random variables are approximated by averaging over the results of a large number of model runs (Krajewski et al., 1991; Abler et al., 1999; Pastres and Ciavatta, 2005; Uhlenbrook and Sieber, 2005; de Kort and Booij, 2007; Krysanova et al., 2007; Raaflaub and Collins, 2006; Roxburgh and Davies, 2006) or by some approximate formula (Skalski et al., 2007). While such methods can accommodate truncated as well as non-normal distributions of random variables, they have some serious disadvantages. For

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larger models, computational requirements to achieve reliable approximations may be excessive; means derived from Monte Carlo simulations can be greatly skewed when a single extreme outcome, though unlikely, nevertheless occurs; and since model uncertainty is dealt with only numerically, there can be no analytical insights. The method we propose and illustrate suffers from none of these limitations.

The particular example which motivates us comes from an environmental economics analysis of greenhouse gas control (Jotzo and Pezzey, 2005). In this analysis there arises the need to calculate a country’s perceived payoff from signing up to a global treaty that limits future emissions of greenhouse gases, where that country’s emissions target is “non-binding” (i.e. optional), and all countries’ future emissions are uncertain. Jotzo and Pezzey assume that once the uncertain future actually arrives, a country enacts a non-binding target only when its perceived net payoff (from selling spare emission permits) is positive. Since the payoff is a linearised function of parameters, and bold, lower-case Latin characters such as \( \mathbf{x}_1, \ldots, \mathbf{x}_n \) denote vectors of fixed dimensions, we use direct integration methods. Also he treated any truncated distribution as a probability distribution over this half-space.

The most investigated case has actually been that of rectangular truncation (Birnbaum and Meyer, 1953; Tallis, 1961; Horrace and Hernandez, 2001; Horrace, 2005a,b), but there is also the paper of Tallis (1965) that deals with the theory of planar truncations. This derived results for multinormal distributions truncated by a number of hyperplanes, and our results overlap Tallis’s to some extent, though they also include some minor extensions. However, Tallis used the method of moment-generating functions (also used in Tallis, 1961), whereas we use direct integration methods. Also he treated any truncated distribution as a probability distribution in its own right (i.e. with a total probability of 1); whereas we treat it as part of a distribution of probability 1 over all of \( \mathbf{R}^n \). This is because in the motivating example, even though values on both sides of the truncating hyperplane occur with non-zero probability, calculating the expected payoff only requires consideration of values on the side of the hyperplane corresponding to profitable outcomes. This difference in approach means our results do not contain Tallis’s normalisation factors.

Our paper therefore has three purposes: to obtain Tallis’s results using an alternative, more direct method, and to give some modest extension to his results, as well as presenting some related truncated multinormal expectation results; to show how all the results can be used in the context of a climate change control treaty; and most importantly, to make all the results more accessible to environmental modellers. In Section 2 we use a well-known theorem on multinormal distributions to prove a specialised truncated expectation result, and then derive the truncated multinormal expectation of both a linear scalar function, and the exponential of a linear scalar function, with respect to an arbitrary truncating hyperplane. Such functions are commonly employed in environmental modelling applications, particularly in hydrology (Keig and McAlpine, 1974; Makhlof and Michel, 1994; Xiong and Guo, 1999; Ferdowsian et al., 2001;) and ecology (Ratnieks, 1996; Pacala et al., 1996; Baguette, 2003). The results follow from Tallis’s work, but are not obvious and therefore hitherto unavailable to a typical environmental modeller. In Section 3 we show how our results can be used in the motivating application of Jotzo and Pezzey (2005) of a climate change control treaty, where a country would make a future, and therefore uncertain, dollar-valued net gain of \( \hat{G} \) from joining the treaty. However, its psychologically perceived “payoff” (used to determine whether or not it chooses to join the treaty) is the net gain plus a term to allow for risk aversion, formally modelled as:

\[
U(\hat{G}) = \hat{G} + s\left(1 - \exp(-r\hat{G})\right)
\]

with \( s, r \) positive constants, where \( s \) is the overall weight of risk aversion and \( r \) is the coefficient of absolute risk aversion. This is a strictly increasing, concave function of \( \hat{G} \) which allows for the diminishing marginal value of money gains. Our results allow the expected payoff \( E[U(\hat{G})] \), to be calculated and compared across countries, which will help to gauge the political feasibility of various types and levels of emission targets.

2. Expectations with respect to truncated multinormal distributions

2.1. Notation and definitions

We use bold, lower-case Greek characters such as \( \alpha = (\alpha_1, \ldots, \alpha_n)^T \) and \( \beta = (\beta_1, \ldots, \beta_n)^T \) to denote vectors of fixed parameters, and bold, lower-case Latin characters such as \( \mathbf{x} = (x_1, \ldots, x_n)^T \) and \( \mathbf{y} = (y_1, \ldots, y_n)^T \) to denote multinormal random variables in the \( n \)-dimensional Euclidean space \( \mathbf{R}^n \).

The standard Euclidean inner product is

\[
\alpha^T \beta = \sum_{j=1}^{n} \alpha_j \beta_j
\]

and the square of the standard Euclidean norm is \( ||\alpha||^2 = \alpha^T \alpha \).

We will assume \( \mathbf{x} \) has zero mean so that its probability density function is given by

\[
\{ \mathbf{x} \in \mathbf{R}^n : x_i \geq c_i, \ldots, x_n \geq c_n \}.
\]

A rectangular truncation excludes all \( n \)-vectors that are not part of the set

\[
\{ x = (x_1, \ldots, x_n)^T \in \mathbf{R}^n : x_1 \geq c_1, \ldots, x_n \geq c_n \}.
\]


\[ \phi_n(x, M) = \frac{\exp\left( -\frac{1}{2}x^T M^{-1} x \right)}{(2\pi)^{n/2} |M|^{1/2}} \]

Here \( M \) is the dispersion (variance–covariance) matrix of the multinormal distribution and \( |M| \) is its determinant. Although we restrict our attention to zero mean multinormal variables, our results could easily be generalised to accommodate a non-zero vector of means. Given a function \( f \) of the random variable \( x \), its expectation, or expected value (over all of \( \mathbb{R}^n \)), is defined as

\[ E[f(x)] = \int f(x) \phi_n(x, M) \, dx \]

If the multinormal distribution is truncated then the integral is taken over the subset of \( \mathbb{R}^n \) defined by the truncating condition.

A scalar function, or functional, is a function whose range lies in the scalar field of real numbers \( \mathbb{R} \). A seminal theorem (Kreyszig, 1978, p. 188) asserts that the action of any bounded linear functional corresponds to an expression of the form \( \alpha^T x \), for some uniquely suitable \( \alpha \in \mathbb{R}^n \). Hence, in what follows, when we speak of a linear functional we are merely referring to an expression of the form \( \alpha^T x \), for some particular \( \alpha \).

2.2. Standard results for multinormal distributions

A characterising feature of the multinormal distribution is its amenability with respect to linear functionals. This is apparent, for example, in the following well-known result (Rohatgi and Ehsanes Saleh, 2001, Section 5.4, Theorem 7).

**Theorem 1.** Let \( x = (x_1, \ldots, x_n)^T \). Then \( x \) has an \( n \)-dimensional normal distribution with zero mean and dispersion matrix \( M \) if and only if every linear combination \( \alpha^T x \) has a univariate normal distribution with zero mean and variance \( \sigma^2 = \alpha^T M \alpha \).

In particular, the above Theorem immediately implies the following result concerning the multinormal expectation of a continuous function whose argument is of the form \((\alpha^T x - c)\), where \( x \) is a multinormal variable.

**Theorem 2.** Let \( U \) be a continuous function and \( x = (x_1, \ldots, x_n)^T \) be a multinormal variable with zero mean and dispersion matrix \( M \). Then

\[ \int_{\mathbb{R}^n} U(\alpha^T x - c) \phi_n(x, M) \, dx = \int_{\mathbb{R}} U(\eta - c) \frac{\exp\left(-\frac{1}{2} \eta^2 \sigma^2 \right)}{\sigma \sqrt{2\pi}} \, d\eta \]

where \( \sigma^2 = \alpha^T M \alpha \).

In essence the result merely indicates that the integrand on the left hand side can be regarded as a function of the scalar \( \alpha^T x \). The multivariate integral has been reduced to a univariate integral, which is much easier to deal with.

2.3. Results for functions of truncated multinormal variables

If \( U: \mathbb{R} \to \mathbb{R} \) is a strictly increasing, continuous function and there exists \( \zeta \in \mathbb{R} \) such that \( U(\zeta - c) = 0 \) then the expression \( U(\alpha^T x - c) \) has associated with it a canonical splitting of \( \mathbb{R}^n \) into two half-spaces,

\[ U^n_c = \{ x \in \mathbb{R}^n : U(\alpha^T x - c) \geq 0 \} \]

\[ U_n^c = \{ x \in \mathbb{R}^n : U(\alpha^T x - c) < 0 \} \]

The separating hyperplane is defined by \( \alpha^T x = \zeta \). For a non-binding target for greenhouse gas emissions, the function \( U \) may describe a country’s payoff from signing up to the target. In this case the zero-payoff hyperplane is the boundary between \( U_n^c \), where the target is enacted, and \( U^n_c \), where the target is not enacted.

Applying Theorem 2 to the subset of continuous functions that are strictly increasing, we obtain the following result. The proof is in Appendix 1.

**Theorem 3.** Suppose that \( U \) is continuous, strictly increasing and that \( U(\zeta - c) = 0 \). Suppose also that \( \alpha \in \mathbb{R}^n \) is an arbitrary, but fixed, vector and that \( x = (x_1, x_2, \ldots, x_n)^T \) is a multinormal random variable with mean zero and dispersion matrix \( M \). If \( \sigma^2 = \alpha^T M \alpha \) then

\[ \int_{U^n_c} U(\alpha^T x - c) \phi_n(x, M) \, dx = \int_{\mathbb{R}} U(\eta - c) \frac{\exp\left(-\frac{1}{2} \eta^2 \sigma^2 \right)}{\sigma \sqrt{2\pi}} \, d\eta. \]

(4)

Note that if \( U \) is a continuous function and there is no \( \zeta \in \mathbb{R} \) for which \( U(\zeta - c) = 0 \), then either \( U > 0 \), which implies that \( U_n^c = \mathbb{R}^n \), or \( U < 0 \), which implies that \( U^n_c = \mathbb{R}^n \). In each of these cases Theorem 3 reverts to Theorem 2.

2.4. Results for arbitrary truncating hyperplanes

The uncertain payoff that one country (say B) will get as a result of country A enacting a non-binding emissions target is given (to a linear approximation) by some function of \( \beta^T x \), for some parameter vector \( \beta \in \mathbb{R}^n \) and some random variable \( x \in \mathbb{R}^n \) which includes various uncertainties in emissions. Country A chooses whether or not to enact the target depending on whether \( \alpha^T x \geq c \) or not (and hence whether or not \( U(\alpha^T x - c) \geq 0 \), if we define payoff \( U(\cdot) \) as a strictly increasing function with \( U(0) = 0 \), as for example in Eq. (2)). But \( \beta^T x \), the uncertain determinant of country B’s payoff, will generally be quite different from \( \alpha^T x \).

The question thus arises whether a result similar to Theorem 3 holds above (or below) some arbitrary hyperplane that is not associated with the argument of the function \( U \). The answer is unfortunately that no simple general expression exists. However, restricting our attention to linear functionals of real multinormal variables, it is possible to derive a simple expression. The following Theorem 4 establishes such a result for
zero mean, independent, unit variance multinormal variables whose dispersion matrix \( M \) is thus the identity matrix. The general case will then follow from Theorem 4. The proof of Theorem 4 progresses as a series of simple steps. Basically, the steps are (i) change coordinates, (ii) integrate the innermost integral using parts, and (iii) use a reduction formula to reduce the integral to the desired expression. Elucidating the steps makes the proof quite lengthy and so it is left to Appendix 2. Theorem 4 and the subsequent Theorem 5 make use of the following notation. We denote the half-space above the hyperplane \( \alpha^T x = c, \, (\alpha \neq 0) \) as

\[
\mathcal{H}_c^\alpha = \{ x \in \mathbb{R}^n : \alpha^T x \geq c \}
\]

We may assume without loss of generality that \( \|\alpha\| = 1 \).

**Theorem 4.** Suppose that \( \alpha, \beta \in \mathbb{R}^n \) are fixed arbitrary vectors, with \( \|\alpha\| = 1 \), and \( x \in \mathbb{R}^n \) is a multinormal random variable such that each \( x_i, i = 1, \ldots, n \) is an independent, normal random variable with zero mean and unit variance. Then

\[
\int_{\mathcal{H}_c^\alpha} \beta^T \phi_a(x, I) \, dx = \exp \left( -\frac{c^2}{2} \right) \frac{\alpha^T \beta}{\sqrt{2\pi}}.
\]  

(5)

We note here that Theorem 4 and the ensuing Theorem 5 could also be established using the mean vector \( \mu \) derived by Tallis (1965), after taking into account the fact that we do not require the truncated distribution to be a probability distribution as Tallis did. As such, the proofs of Theorem 4 and Theorem 5 serve as a conceptually simpler, alternative derivation of Tallis’s mean vector that avoids the need for the moment-generating function. The proof of Theorem 5 is in Appendix 3.

**Theorem 5.** Suppose that \( \alpha, \beta \in \mathbb{R}^n \) are fixed, arbitrary vectors and \( x \in \mathbb{R}^n \) is a zero mean multivariate normal random variable with dispersion matrix \( M \). Then

\[
\int_{\mathcal{H}_c^\alpha} \beta^T x \phi_a(x, M) \, dx = \exp \left( -\frac{c^2}{2\sigma^2} \right) \frac{\alpha^T M \beta}{\sigma \sqrt{2\pi}}
\]  

(6)

where \( \sigma^2 = \alpha^T M \alpha \).

We may also obtain a simple expression for the expectation of the exponential of a linear functional. Such an expectation is, modulo a normalising factor, the moment-generating function, and hence the result is in Tallis (1965), but for completeness we include the result here and provide the proof in Appendix 4.

**Theorem 6.** Suppose that \( \alpha, \beta \in \mathbb{R}^n \) are fixed, arbitrary vectors and \( x \in \mathbb{R}^n \) is a zero mean multivariate normal random variable with dispersion matrix \( M \). Then

\[
\int_{\mathcal{H}_c^\alpha} \exp (r \beta^T x) \phi_a(x, M) \, dx = \exp \left( \frac{1}{2} r^2 \beta^T M \beta \right) \frac{\phi_\sigma \left( r \alpha^T M \beta - c \right)}{\sigma \sqrt{2\pi}}
\]  

(7)

where \( \phi_\sigma [z] = 1/\sqrt{2\pi} \int_{-\infty}^{z} \exp \left( -1/2 \sigma^2 s^2 \right) ds \) is the cumulative distribution function of the univariate normal distribution, and \( \sigma^2 = \alpha^T M \alpha \).

3. Example from a climate change treaty with uncertainty in emissions: quantifying risk aversion

Here we calculate the expected payoffs for two different countries, A and B, that result from country A signing up to a climate change treaty by accepting a non-binding target for its future greenhouse gas emissions, given that the payoff function is \( U(\bar{G}) = \hat{G} + s(1 - \exp(-r\hat{G})) \) as in Eq. (2). Because the target is non-binding, signing up to the target now does not mean that it will be enacted when an uncertain future arrives. It is precisely because of country A’s option to enact the target when the future arrives that the expected payoff to both countries are defined over truncated half-spaces. The simplified linear model of Jotzo and Pezzey (2005) shows how the expected payoffs can be defined in terms of vectors \( \alpha \) and \( \beta \) and parameters \( s_A, c_A, s_B, c_B \) and \( r \) as follows, and we show in appendices how to calculate the expected payoffs stated using the above Theorems.

3.1. Expected payoff to country A from A signing up to a non-binding target

Assuming that the dollar-valued net gain, \( \bar{G}_A \), of country A is a linearised function of the multinormal random variable \( x \) (the vector of proportional emission targets for all countries), we have \( \bar{G}_A = \alpha^T x - c_A \). According to Eq. (2) the payoff to country A is given by

\[
U(\bar{G}_A) = \hat{G}_A + s_A \left( 1 - \exp(-r\hat{G}_A) \right)
\]  

(8)

where we suppose that \( s_A > 0 \), and \( r > 0 \). The expected payoff to country A from A signing up to a non-binding emissions target is defined as

\[
E[U(\bar{G}_A)|\lambda] = \int_{\mathcal{H}_{s_A>0}} \left[ \hat{G}_A + s_A \left( 1 - \exp(-r\hat{G}_A) \right) \right] \phi_\sigma(x, M) \, dx
\]

Applying Theorem 3 we obtain the result (see Appendix 5 for proof):

\[
E[U(\bar{G}_A)|\lambda] = (s_A - c_A) \Phi \left( \frac{-c_A}{\sigma} \right)
\]  

\[
- s_A \exp \left( \frac{r c_A}{2} \right) \Phi \left( \frac{-c_A + r \sigma^2}{2\sigma^2} \right) + \frac{\sigma}{\sqrt{2\pi}} \exp \left( \frac{-c_A^2}{2\sigma^2} \right)
\]  

(9)

with \( \sigma^2 = \alpha^T M \alpha \), where the three terms in Eq. (9) correspond to those in Eq. (8).
3.2. Expected payoff to country B from A signing up to a non-binding target

Taking $\tilde{G}_A$ and $r$ as before, and setting $G_B = \beta^T x - c_B$ and $s_B > 0$, the expected payoff to country B from A signing up to a non-binding emissions target is defined as

$$E[U(G_B)|A] = \int_{\tilde{G}_A > 0} [G_B + s_B (1 - \exp(-r \tilde{G}_B))] \phi_a(x,M) dx$$

$$= \int_{G_B \geq 0} [(s_B - c_B) - s_B \exp(-r \tilde{G}_B) + \beta^T x]$$

$$\times \phi_a(x,M) dx$$

(10)

Theorems 5 and 6 can be utilised to obtain the result (see Appendix 6 for proof):

$$E[U(G_B)|A] = (s_B - c_B) \Phi \left( -\frac{c_B}{\sigma} \right)$$

$$- s_B \exp \left( r c_B + \frac{1}{2} \beta^T M \beta \right) \Phi \left( \frac{-c_A + r \alpha^T M \beta}{\sigma} \right)$$

$$+ \frac{\alpha^T M \beta}{\sqrt{2 \pi \sigma}} \exp \left( -\frac{c_A^2}{2 \sigma^2} \right)$$

(11)

with $\alpha^2 = \alpha^T M \alpha$, and where the three terms in Eq. (11) correspond to the three terms in the integrand of Eq. (10).

These results highlight the effect of risk aversion that results from the strict concavity of the payoff function Eq. (2). The parameter $r$ describes the sharpness of a country’s risk aversion, and $s_A$ and $s_B$ its respective importance in comparison to the dollar-valued gains $G_A$ and $G_B$ themselves. So setting $r = s_A = s_B = 0$ in the above results, and subtracting the outcomes from the original results reveals the net effects of risk aversion.

An illustration of the use of results Eqs. (9) and (11) comes from Jotzo and Pezzey’s (2005) work on non-binding targets for greenhouse gas emissions. For a draft version of the manuscript, and Frank Jotzo for many helpful discussions about modelling economic alternatives for greenhouse gas abatement. We are also indebted to Tony Jakeman and an anonymous referee for helpful suggestions.

4. Conclusion

The results presented here provide a way of calculating the expected values of some of the functions commonly used in environmental modelling, when the underlying distribution is taken to be multivariate normal, subject to a truncation or cut-off condition. Cut-off conditions arise naturally in many instances in environmental and other sciences and the associated variables can often be characterised as being approximately normally distributed. For example, results such as Theorems 5 and 6 are in applications dealing with genetic selection of plants and animals through independent culling levels (Tallis, 1961, 1965), personnel selection (Birnbaum and Meyer, 1953), rainfall estimation and modelling (Hutchinson et al., 1993), multivariate regression models arising from household economic surveys (Tobin, 1958; Amenti, 1973, 1974) and in conjunction with other methods to model technical efficiency of electric utility plants (Horrace, 2005a,b). The formulae presented in this paper should therefore be widely applicable. As well as allowing efficient estimation of mean values of functions encountered in modelling applications, the results also provide theoretical insight into the nature of uncertainties associated with particular modelling approaches. It is our hope that by enhancing the accessibility of such results and illustrating their use with a concrete example, environmental modellers will find it easier to use them in their own applications when appropriate.

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Appendices

In this section we present mathematical proofs for the results cited above.

Appendix 1. Proof of Theorem 3. Define the continuous function $U$ as

$$\tilde{U}(z) = \begin{cases} U(z) & \text{if } z \geq \zeta - c \\
0 & \text{if } z < \zeta - c \end{cases}$$

Note that since $U$ is strictly increasing, $\tilde{U} \geq 0$. From Theorem 2 we therefore have
\[
\int U(\alpha^T x - c) \phi_n(x) \, dx = \int \tilde{U}(\alpha^T x - c) \phi_n(x, M) \, dx
\]

Employing the change of coordinates \( y = x - c \alpha \) we find that

\[
(2\pi)^{n/2} J = \exp \left( -\frac{c^2}{2} \right) \int_{\mathbb{R}^n} \beta^T (y + c\alpha) \exp \left[ -\frac{1}{2}(y^T y + 2c\alpha^T y) \right] \, dy
\]

\[
= \exp \left( -\frac{c^2}{2} \right) \sum_{j=1}^{n} J_j
\]

where

\[
J_j = \int_{\mathbb{R}^{n-1}} \beta_j(y_j + c\alpha) \exp \left[ -\frac{1}{2} \sum_{i=1}^{n-1} (y_i^2 + 2c\alpha_i y_i) \right] \, dy_1 \cdots \, dy_{n-1}
\]

and where we have set \( \xi = -1/\alpha_n \sum_{i=1}^{n-1} \alpha_i y_i \). Hence if \( j \neq n \) then

\[
J_j = \int_{\mathbb{R}^{n-1}} A_n \beta_j(y_j + c\alpha) \exp \left[ -\frac{1}{2} \sum_{i=1}^{n-1} (y_i^2 + 2c\alpha_i y_i) \right] \, dy_1 \cdots \, dy_{n-1}
\]

where

\[
A_n = \int_{\xi} \exp \left[ \frac{1}{2} (y_n^2 + 2c\alpha_n y_n) \right] \, dy_n.
\]

To simplify the expression for \( J_j \) we perform an integration by parts. By the Fundamental Theorem of Calculus we have

\[
dA_n = \frac{\alpha_j}{\alpha_n} \exp \left\{ -\frac{1}{2} \left[ \frac{1}{\alpha_n} \left( \sum_{i=1}^{n-1} \alpha_i y_i \right)^2 - 2c \sum_{i=1}^{n-1} \alpha_i y_i \right] \right\} \, dy_j
\]

and that if

\[
dv = \beta_j(y_j + c\alpha_j) \exp \left[ -\frac{1}{2} (y_j^2 + 2c\alpha_j y_j) \right] \, dy_j
\]

then

\[
v = -\beta_j \exp \left[ -\frac{1}{2} (y_j^2 + 2c\alpha_j y_j) \right].
\]

The integration by parts formula reads

\[
\int_{\mathbb{R}} A_n \, dv = [A_n v]_{-\infty}^{\infty} - \int_{\mathbb{R}} v \, dA_n
\]

and so integrating by parts, noting that \( A_n v \to 0 \) as \( y_j \to \pm \infty \), we get

\[
J_j = \int_{\mathbb{R}^{n-1}} \frac{\alpha_j \beta_j}{\alpha_n} \exp \left[ -\frac{1}{2} \left( \sum_{i=1}^{n-1} y_i^2 + \frac{1}{\alpha_n} \left( \sum_{i=1}^{n-1} \alpha_i y_i \right)^2 \right) \right] \, dy_1 \cdots \, dy_{n-1}
\]

Conversely, if \( j = n \) then

\[
J_n = \int_{\mathbb{R}^{n-1}} \beta_n \exp \left[ -\frac{1}{2} \sum_{i=1}^{n-1} (y_i^2 + 2c\alpha_i y_i) \right] \, dy_1 \cdots \, dy_{n-1},
\]

where

\[
C_n = \int_{\xi} \exp \left[ -\frac{1}{2} \left( \sum_{i=1}^{n} y_i^2 + 2c\alpha_n y_n \right) \right] \, dy_n
\]

Substituting this expression for \( C_n \) into the expression for \( J_n \) gives \( J_n = \beta_n J^n \), which in turn implies that

\[
(2\pi)^{n/2} J = \exp \left( -\frac{c^2}{2} \right) \sum_{j=1}^{n} \frac{\alpha_j \beta_j}{\alpha_n} J^n
\]

The integral \( J^n \) can now be reduced to a simple expression by means of a reduction formula. Considering the exponent in the integrand we may deduce that

\[
\sum_{i=1}^{n-1} y_i^2 + \frac{1}{\alpha_n} \left( \sum_{i=1}^{n-1} \alpha_i y_i \right)^2
\]

\[
= \frac{1}{\alpha_n^2} \left( y_1 \sqrt{\alpha_2 + \alpha_1^2} + \alpha_1 \sum_{i=2}^{n-2} \alpha_i y_i \right)^2
\]

\[
+ \sum_{i=2}^{n-1} y_i^2 + \frac{1}{\alpha_n^2} \left( \sum_{i=2}^{n-1} \alpha_i y_i \right)^2.
\]

Letting
\[ t = y_1 \sqrt{\alpha_n^2 + \alpha_1^2} + \frac{\alpha_1 \sum_{i=2}^{n-1} \alpha_i y_i}{\sqrt{\alpha_n^2 + \alpha_1^2}} \]

we have \( dt = \sqrt{\alpha_n^2 + \alpha_1^2} dy_1 \) and so

\[ J^* = \int_{\mathbb{R}^{n-2}} \exp \left\{ -\frac{1}{2} \left[ \sum_{i=2}^{n-1} y_i^2 + \frac{1}{\alpha_n^2 + \alpha_1^2} \left( \sum_{i=2}^{n-1} \alpha_i y_i \right)^2 \right] \right\} \]

\[ = \exp \left( -\frac{1}{2} \frac{1}{\alpha_n^2 + \alpha_1^2} \right) \int_{\mathbb{R}^{n-2}} \exp \left\{ -\frac{1}{2} \left[ \sum_{i=2}^{n-1} y_i^2 + \frac{1}{\alpha_n^2 + \alpha_1^2} \left( \sum_{i=2}^{n-1} \alpha_i y_i \right)^2 \right] \right\} dy^*
\]

where we have used \( dy^* \) as shorthand for \( dy_2 \cdots dy_{n-1} \). Iterating this result we obtain

\[ J^* = \alpha_n (2\pi)^{(n-1)/2}. \]

Substituting this back into the expression for \( (2\pi)^{n/2} J \), cancelling like terms, we find

\[ J = \exp \left( -\frac{c^2}{2} \right) \frac{\alpha^T \beta}{\sqrt{\alpha^T \alpha}} \]

and the proof is complete. \( \square \)

**Appendix 3. Proof of Theorem 5.** For convenience let

\[ J = \int_{\mathbb{R}^n} \beta^T x \phi_n(x,M) dx \]

so that

\[ (2\pi)^{n/2} |M|^{1/2} J = \int_{\mathbb{R}^n} \beta^T x \exp \left( -\frac{1}{2} x^T M^{-1} x \right) dx_1 \cdots dx_n. \]

The matrix \( M^{-1} \) is by definition a symmetric, positive definite matrix and so possesses a Cholesky decomposition \( M^{-1} = \Psi^T \Psi \), where \( \Psi \), the Cholesky factor, is a nonsingular, upper triangular matrix (Gentle, 1998). Transforming the variables to \( y = \Psi x \) we find that \( dy = |\Psi| dx \). Note that \( M^{-1} = \Psi^T \Psi \) implies that \( |M| = |\Psi|^{-2} \) and so we have

\[ (2\pi)^{n/2} J = \int_{\mathbb{R}^n} \tilde{\beta}^T y \exp \left( -\frac{1}{2} y^T \Psi^{-1} \Psi y \right) dy \]

where we have set \( \tilde{\beta} = \Psi^{-T} \beta \) and

\[ \mathcal{H}_c^\circ = \{ y \in \mathbb{R}^n : \tilde{\alpha}^T y \geq c \} \quad (12) \]

where \( \tilde{\alpha} = \Psi^{-T} \alpha \). Applying Theorem 4 we find that

\[ J = \exp \left( -\frac{1}{2} \frac{c^2}{ \| \alpha \| } \right) \frac{\tilde{\alpha}^T \tilde{\beta}}{\sqrt{2\pi \| \alpha \| }} = \exp \left( -\frac{1}{2} \frac{c^2}{\sigma^2} \right) \frac{\alpha^T M \beta}{\sigma \sqrt{2\pi}} \]

with \( \sigma = \alpha^T M \alpha \) \( \square \)

**Appendix 4. Proof of Theorem 6.** Let

\[ J = \int_{\mathbb{R}^n} \exp (r \beta^T x) \phi_n(x,M) dx \]

Introducing the Cholesky decomposition of \( M^{-1} = \Psi^T \Psi \) and transforming the variables to \( y = \Psi x \), as was done in Appendix 3, we find that

\[ (2\pi)^{n/2} |M|^{1/2} \Psi^T J = \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} y^T y - 2 \beta^T y \right) dy, \]

where \( \mathcal{H}_c^\circ \) is defined in Appendix 3 and \( \tilde{\beta} = r \Psi^{-1} \beta \). Setting \( z = y - \tilde{\beta} \) it follows that \( y^T y - 2 \beta^T y = z^T z - \| \beta \|^2 \). Hence if we define \( \rho = c - \alpha^T \beta \) we have

\[ (2\pi)^{n/2} |M|^{1/2} \Psi^T J = \exp \left( \frac{1}{2} \| \beta \|^2 \right) \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} z^T z \right) dz \]

Following the method of Tallis (1965), we let \( z = B w \), where the first column of the matrix \( B \) is the unit vector \( \tilde{\alpha} \) and \( \| \alpha \| \) and the other columns of \( B \) form an orthonormal basis for the hyperplane defined by \( \tilde{\alpha}^T z = \tilde{\rho} \). Then

\[ \mathcal{H}_p = \{ w \in \mathbb{R}^n : \tilde{\alpha}^T B w \geq \tilde{\rho} \} \]

where \( \tilde{\rho} = \rho / \| \tilde{\alpha} \| \). Therefore, setting \( dw^* = dw_2 \cdots \cdot dw_n \) and separating the integral accordingly, we have

\[ (2\pi)^{n/2} |M|^{1/2} \Psi^T J = \exp \left( \frac{1}{2} \| \beta \|^2 \right) \int_{\mathbb{R}^{n-1}} \exp \left( -\frac{1}{2} \sum_{i=2}^{n} w_i^2 \right) \]

\[ \times \int_{\mathcal{H}_p} \exp \left( -\frac{1}{2} w_1^2 \right) dw_1 dw^* \]

\[ = \exp \left( \frac{1}{2} \| \beta \|^2 \right) (2\pi)^{(n-1)/2} \]

\[ \times \int_{\mathcal{H}_p} \exp \left( -\frac{1}{2} w_1^2 \right) dw_1 \]

\[ = \exp \left( \frac{1}{2} \| \beta \|^2 \right) (2\pi)^{n/2} \Phi \left( -\tilde{\rho} \right) \]

Cancelling like terms and substituting for \( \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{\rho} \) we obtain Eq. (7) as required. \( \square \)
Appendix 5. Derivation of Eq. (9). The result follows from
Theorem 3. We note that \( U(z) = z + s_A (1 - \exp(-rz)) \) is a continuous function that is increasing with \( z \), satisfying \( U(0) = 0 \). Noting also that \( G_A \geq 0 \) defines the half-space
\[
\mathcal{H}^c_A = \{ x \in \mathbb{R}^d : U(a^T x - c_A) \geq 0 \},
\]
we wish to calculate
\[
E[U(G_B|A)] = \int \left[ G_B + s_B (1 - \exp(-r(G_B))) \right] \phi_u(x, M) \, dx
\]
with \( G_B = \beta^T x - c_B \). We can write \( E[U(G_B|A)] = \phi_b(x, M) \, dx \)
where
\[
I_{B_1} = \int \phi_b(x, M) \, dx
\]
\[
I_{B_2} = \int \beta^T \phi_b(x, M) \, dx
\]
\[
I_{B_3} = \int \exp(-r\beta^T x) \phi_b(x, M) \, dx
\]

\( I_{B_1} \) may be evaluated by first completing the square and then using the method of Tallis (1965) to transform the coordinates, as was done in Appendix 4. This results in
\[
I_{B_1} = \Phi \left[ \frac{-c_A}{\sigma} \right]
\]
with \( \sigma^2 = \alpha^T M \alpha \). Theorems 5 and 6, respectively, imply that
\[
I_{B_2} = \frac{\alpha^T M \beta}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2} \frac{c_B^2}{\sigma^2} \right)
\]
\[
I_{B_3} = \exp \left( r c_B + \frac{1}{2} \beta^T M \beta \right) \Phi \left[ \frac{-c_A + r \alpha^T M \beta}{\sigma} \right]
\]
Combining the above expressions for \( I_{B_1}, I_{B_2}, I_{B_3} \), we obtain Eq. (11). \( \square \)

References

Amemiya, T., 1974. Multivariate regression and simultaneous equation models when the dependent variables are truncated normal. Econometrica 42, 999–1012.


